

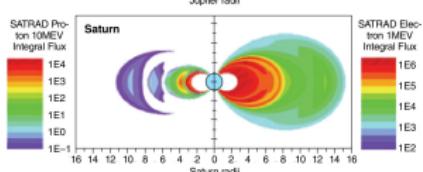
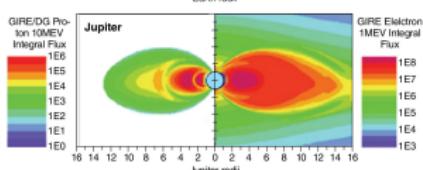
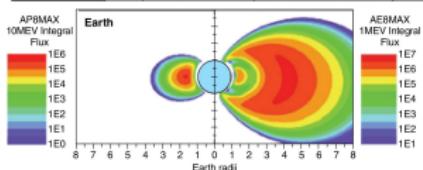
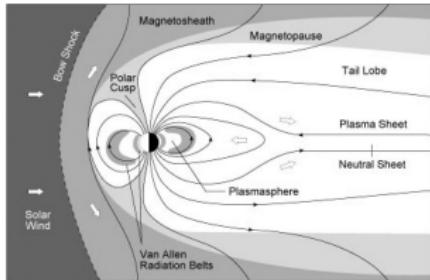
Charged Particle Dynamics in Planetary Magnetic Fields

Patrick Guio

University College London

COMAC 2018, Chiang Mai, 3 July 2018

Background and Context



Trapped Charged Particles

Radiation belts

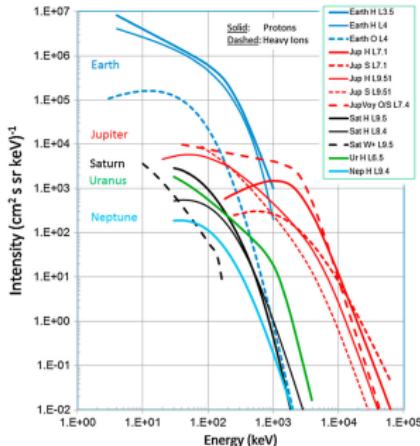
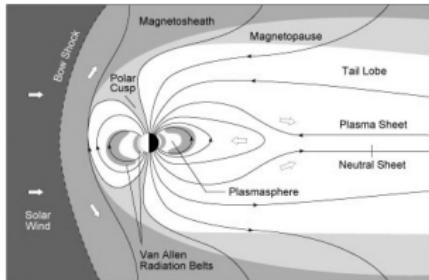
- Particles e^- and p^+
- Energy $\simeq 10s \text{ keV} - 100s \text{ MeV}$
- Distance $\lesssim \text{few } R_P$

Ring currents

- Particles e^- , p^+ and heavy ions
- Energy $\simeq 10s \text{ keV}$
- Distance $\gtrsim \text{few } R_P \text{ and } \lesssim \text{tens } R_P$

Baker [2014]

Background and Context



Mauk [2014]

Trapped Charged Particles

Radiation belts

- Particles e^- and p^+
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Ring currents

- Particles e^- , p^+ and heavy ions
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Single Charged Particle Motion

- Newton's law on particle mass m and charge q
- Lorentz force with **imposed** external magnetic field \mathbf{B}

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}$$

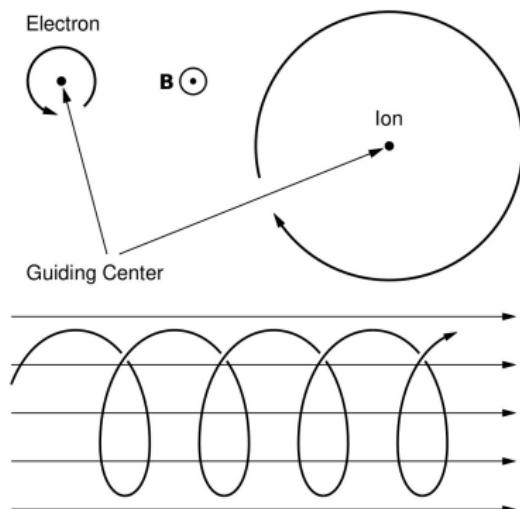
$$m \frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}(\mathbf{r})$$

$$\mathbf{r}(t=0) = \mathbf{r}_0$$

$$\mathbf{v}(t=0) = \mathbf{v}_0 \quad \iff \quad (\text{energy, direction})_0$$

- Lorentz force \perp to both \mathbf{v} and \mathbf{B}
- \implies kinetic energy $W = \frac{1}{2}mv^2$ conserved.
- Particles move on helices \parallel to \mathbf{B} .
- Relativistic formulation m replaced by $m_0\sqrt{1 - v^2/c^2}$

Uniform Field: Gyro and Helicoidal Motions



- Gyro (cyclotron) frequency

$$\Omega_g = \frac{qB}{m}$$

- Gyro (cyclotron or Larmor) radius

$$r_g = \frac{mv_{\perp}}{qB}$$

- Pitch angle α

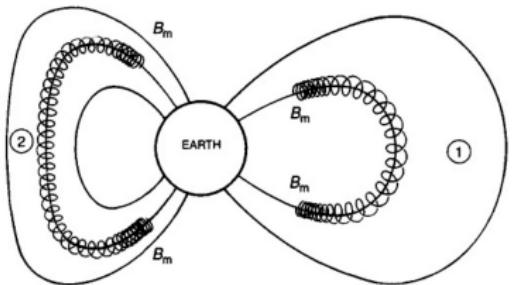
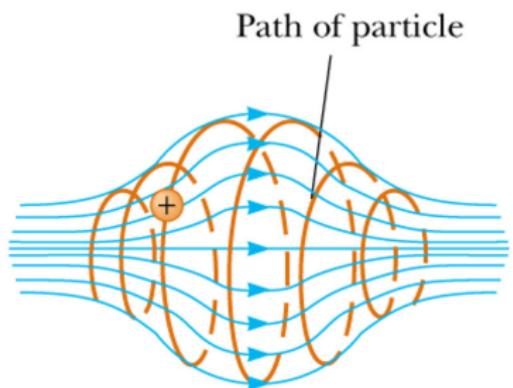
$$\tan(\alpha) = \frac{v_{\perp}}{v_{\parallel}}$$

- Magnetic moment

$$\mu = \frac{mv_{\perp}^2}{2B} = \frac{W_{\perp}}{B} = Ia$$

Baumjohann and Treumann
[1996]

Inhomogeneous Magnetic Field: Bounce Motion



- $(\mu \propto v_{\perp}^2/B)$ 1st invariant is conserved

$$\Rightarrow \frac{\sin^2(\alpha_{\text{eq}})}{B(R_{\text{eq}}, 0)} = \frac{\sin^2(\alpha)}{B(R, \lambda)}$$

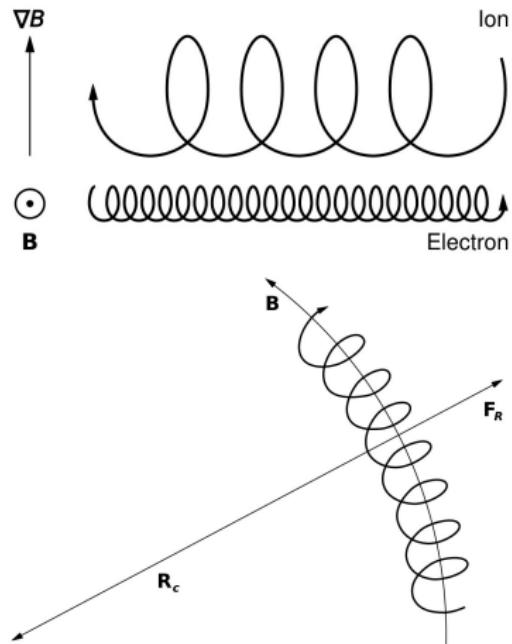
- **Mirror point:** when $\alpha = \pi/2$, latitude λ_m such

$$\sin^2(\alpha_{\text{eq}}) = \frac{B(R_{\text{eq}}, 0)}{B(r_m, \lambda_m)}$$

- **Loss cone:** pitch angle α_{LC}

$$\sin^2(\alpha_{\text{LC}}) = \frac{B(R_{\text{eq}}, 0)}{B(R_p, \lambda_p)}$$

Magnetic Drifts: Drift Motion



- First-Order Approximation

$$r_g \ll |B/\nabla B|$$

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 + (\mathbf{r} \cdot \nabla) \mathbf{B}_0$$

- Gradient drift (∇B)

$$\mathbf{v}_g = \frac{mv_{\perp}^2}{2q} \frac{\mathbf{B} \times \nabla B}{B^3}$$

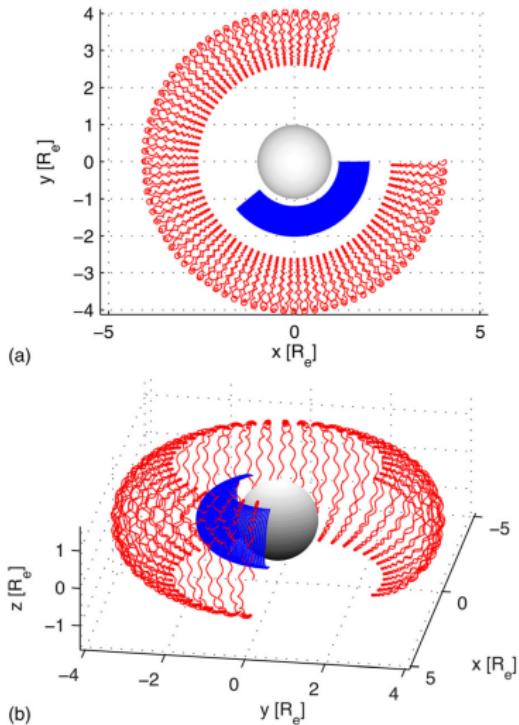
- Curvature drift

$$\mathbf{v}_c = \frac{mv_{\parallel}^2}{q} \frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2}$$

where \mathbf{R}_c is the local radius of curvature

Baumjohann and Treumann
[1996]

Charged Particle Motion Tracing



Öztürk [2012]

- 10 MeV protons in the Earth's dipole field
- Initial conditions
 - Position
 $R_{\text{eq}} = (2R_P, 0, 0)$ and
 $R_{\text{eq}} = (4R_P, 0, 0)$
 - Pitch angle $\alpha_{\text{eq}} = 30 \text{ deg}$
- dipole in the $-\hat{z}$ direction
- Characteristic bounce and drift motions
 - $2R_P$
 $\tau_c = 0.02 \text{ s}$, $\tau_b = 1.7 \text{ s}$,
 $\tau_d = 332 \text{ s}$
 - $4R_P$
 $\tau_c = 0.13 \text{ s}$, $\tau_b = 3.4 \text{ s}$,
 $\tau_d = 166 \text{ s}$

Bounce and Drift Periods Integrals

For a particle **trapped** between $\pm \lambda_m$

- Bounce period

$$\tau_b = \int_0^{\tau_b} dt = 4 \int_0^{s_m} \frac{ds}{v_{\parallel}} = 4 \int_0^{\lambda_m} \frac{ds}{d\lambda} \frac{d\lambda}{v_{\parallel}}$$

where

$$v_{\parallel} = v \left(1 - \frac{B(s)}{B_{\text{eq}}} \sin^2 \alpha_{\text{eq}} \right)^{1/2}$$

- Drift period

$$\tau_d = \frac{2\pi}{\Delta\phi} \tau_b,$$

with $\Delta\phi$ azimuthal drift for τ_b

$$\Delta\phi = \int_0^{\tau_b} \frac{v_D dt}{r \cos \lambda} = 4 \int_0^{\lambda_m} \frac{v_D}{r \cos \lambda} \frac{ds}{v_{\parallel}},$$

$$v_D = v_g + v_c \quad \Rightarrow \quad \Delta\phi = \Delta\phi_g + \Delta\phi_c$$

Adiabatic Invariants

$r_g \ll |B/\nabla B|$ and no time dependence

- First adiabatic invariant: magnetic moment

$$\mu = \frac{mv_{\perp}^2}{2B} = \frac{W_{\perp}}{B}$$

- Second adiabatic invariant: longitudinal invariant

$$J = \oint mv_{\parallel} ds = 2m\mathcal{L} \langle v_{\parallel} \rangle$$

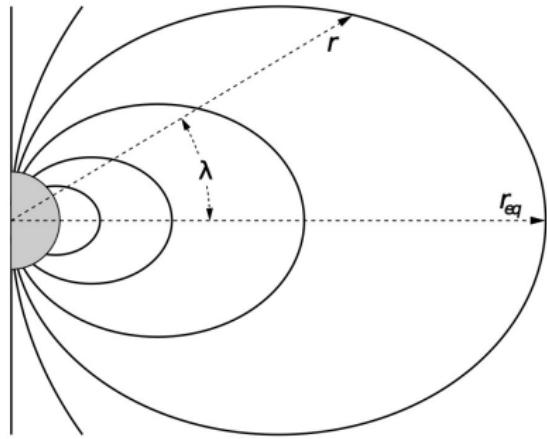
where \mathcal{L} is the length of field line between mirror points.

- Third adiabatic invariant: drift invariant

$$\phi = \oint_D v_D r d\psi = \frac{2\pi m}{q^2} \mathcal{M}$$

where \mathcal{M} is the magnetic moment of the axisymmetric field.

Dipole Field



- Magnitude B

$$B(r, \lambda) = \frac{B_P R_P^3}{r^3} \sqrt{1 + 3 \sin^2 \lambda}$$

- Dipole field line equation

$$r(\lambda) = R_{\text{eq}} \cos^2 \lambda$$

- $R_{\text{eq}} \equiv L \implies L\text{-shell}$

$$B(L, \lambda) = \frac{B_P}{L^3} \frac{\sqrt{1 + 3 \sin^2 \lambda}}{\cos^6 \lambda}$$

Baumjohann and Treumann
[1996]

- B symmetric in azimuth

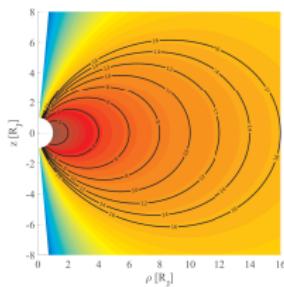
$$B(r, \lambda) = \frac{\mu_0 \mathcal{M}}{4\pi r^3} [-2 \sin \lambda \hat{r} + \cos \lambda \hat{\lambda}]$$

- $\mathcal{M}_J \sim 20000 \mathcal{M}_E$,
 $\mathcal{M}_S \sim 600 \mathcal{M}_E$

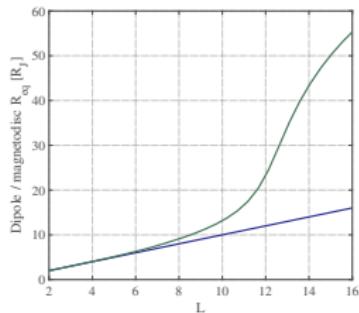
- $B_J \sim 14 B_E$, $B_S \sim 0.7 B_E$

Dipole Field vs. Magnetodisc Field

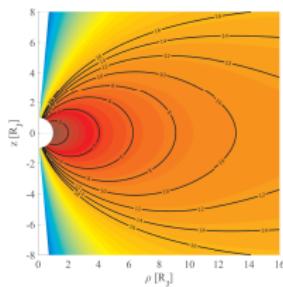
- UCL Magnetodisc for Jupiter with magnetopause at $R_{\text{mp}} = 90R_J$ and hot plasma index $K_h = 3 \cdot 10^7 \text{ Pa m T}^{-1}$
- Field line \iff Euler potential isocontour $\alpha = \text{constant}$



Jupiter internal
(dipole) field



Equivalent dipole L
parameter



Jupiter internal and
magnetodisc field

Bounce Integral for Dipole Field

- Guiding centre path \iff field line, 1st invariant conserved
- Mirror point latitude $\lambda_m(\alpha_{\text{eq}})$

$$\sin^2 \alpha_{\text{eq}} = \frac{B_{\text{eq}}}{B_m} = \frac{\cos^6 \lambda_m}{\sqrt{1 + 3 \sin^2 \lambda_m}}$$

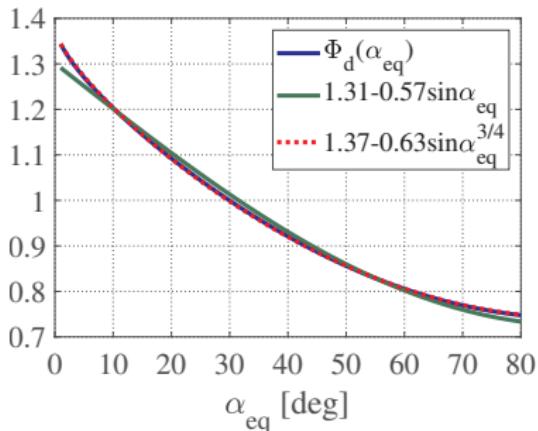
- Bounce period τ_b^d

$$\tau_b^d = \frac{4R_{\text{eq}}}{v} \Phi_d(\alpha_{\text{eq}}) = LR_{\text{P}} \frac{2(2m)^{1/2}}{W^{1/2}} \Phi_d(\alpha_{\text{eq}}),$$

where Φ_d is dimensionless function of geometry and α_{eq}

$$\Phi_d(\alpha_{\text{eq}}) = \int_0^{\lambda_m(\alpha_{\text{eq}})} \frac{\cos \lambda \sqrt{1 + 3 \sin^2 \lambda}}{\left[1 - \frac{\sqrt{1+3 \sin^2 \lambda}}{\cos^6 \lambda} \sin^2 \alpha_{\text{eq}} \right]^{\frac{1}{2}}} d\lambda$$

Bounce Integral Approximation for Dipole Field



- Polynomial Approximation

$$\tau_b^d \sim LR_P \frac{2(2m)^{1/2}}{W^{1/2}} (1.31 - 0.57 \sin \alpha_{\text{eq}})$$

- Alternative expression

$$\tau_b^d \sim LR_P \frac{2(2m)^{1/2}}{W^{1/2}} (1.37 - 0.63 (\sin \alpha_{\text{eq}})^{3/4})$$

Drift Integral for Dipole Field

- Curl free field $\nabla \times \mathbf{B} = \mathbf{0} \implies \nabla B = -BR_c/R_c^2$ and

$$\mathbf{v}_c = \frac{mv_{\parallel}^2}{q} \frac{\mathbf{B} \times \nabla B}{B^3}$$

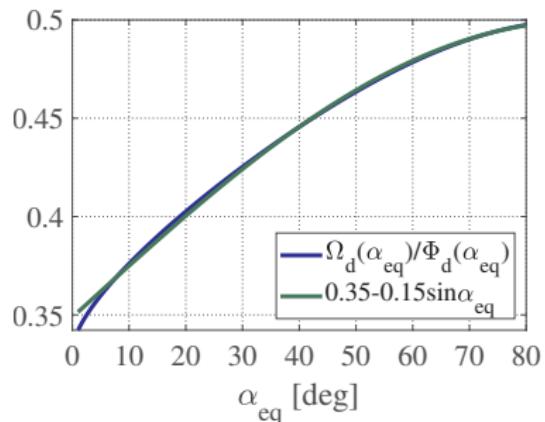
- Drift period

$$\tau_d^d = \frac{2\pi}{\Delta\phi} \tau_b^d = \frac{\pi q B_P R_P^2}{3LW} \frac{\Phi_d(\alpha_{\text{eq}})}{\Omega_d(\alpha_{\text{eq}})},$$

where Ω_d is dimensionless function of geometry and α_{eq}

$$\begin{aligned} \Omega_d(\alpha_{\text{eq}}) &= \int_0^{\lambda_m(\alpha_{\text{eq}})} \left[1 - \frac{\sqrt{1 + 3 \sin^2 \lambda}}{2 \cos^6 \lambda} \sin^2 \alpha_{\text{eq}} \right] \\ &\quad \left[1 - \frac{\sqrt{1 + 3 \sin^2 \lambda}}{\cos^6 \lambda} \sin^2 \alpha_{\text{eq}} \right]^{-\frac{1}{2}} \frac{\cos^3 \lambda (1 + \sin^2 \lambda)}{(1 + 3 \sin^2 \lambda)^{3/2}} d\lambda \end{aligned}$$

Drift Integral Approximation for Dipole Field



■ Polynomial Approximation

$$\tau_d^d \sim \frac{\pi q B_P R_P^2}{3LW} \frac{1}{0.35 + 0.15 \sin \alpha_{\text{eq}}}$$

Particle Tracing Pusher: Boris Algorithm (1970)

- Explicit in time \Rightarrow fast
- Time-centred \Rightarrow 2nd order accurate
- ‘Leap-frog’ (requires $v_{-1/2}$)

$$(\mathbf{r}_k, \mathbf{v}_{k-\frac{1}{2}}) \xrightarrow{t+\Delta t} (\mathbf{r}_{k+1}, \mathbf{v}_{k+\frac{1}{2}})$$

- Phase space conserved
- Boris algorithm steps

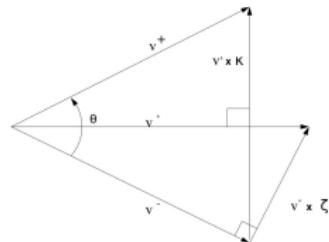
$$1) \quad \mathbf{v}^- = \mathbf{v}_{k-\frac{1}{2}}$$

$$2) \quad \mathbf{v}' = \mathbf{v}^- + \mathbf{v}^- \times \boldsymbol{\zeta}$$

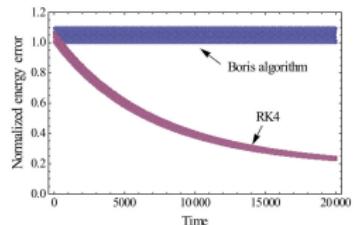
$$3) \quad \mathbf{v}^+ = \mathbf{v}^- + \mathbf{v}' \times \boldsymbol{\kappa}$$

$$4) \quad \mathbf{v}_{k+\frac{1}{2}} = \mathbf{v}^+$$

$$5) \quad \mathbf{r}_{k+1} = \mathbf{r}_k + \mathbf{v}_{k+\frac{1}{2}} \Delta t$$



$$\tan \frac{\theta}{2} = \frac{qB}{m} \frac{\Delta t}{2}$$
$$\zeta = \frac{qB_k}{m} \frac{\Delta t}{2} \quad \text{and} \quad \kappa = \frac{2\zeta}{1 + \zeta^2}$$



Qin et al. [2013]

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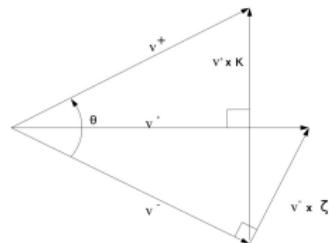
$$1) \quad \mathbf{v}^- = \mathbf{v}_{k-\frac{1}{2}} + \frac{q\mathbf{E}_k}{m} \frac{\Delta t}{2}$$

$$2) \quad \mathbf{v}' = \mathbf{v}^- + \mathbf{v}^- \times \boldsymbol{\zeta}$$

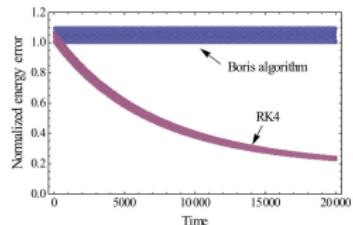
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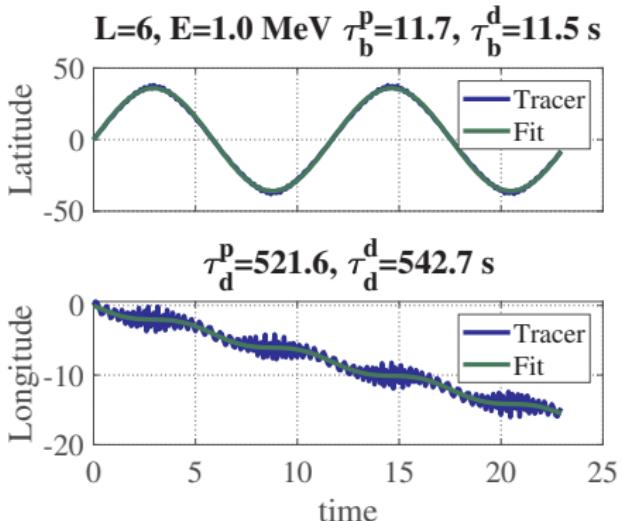


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$$\zeta = \frac{qB_k}{m} \frac{\Delta t}{2} \quad \text{and} \quad \kappa = \frac{2\zeta}{1 + \zeta^2}$$



Qin et al. [2013]

Particle Tracing: Latitude and Longitude Fitting



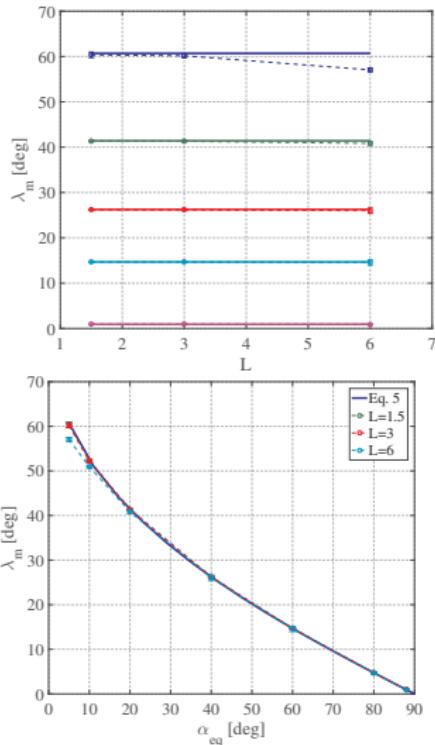
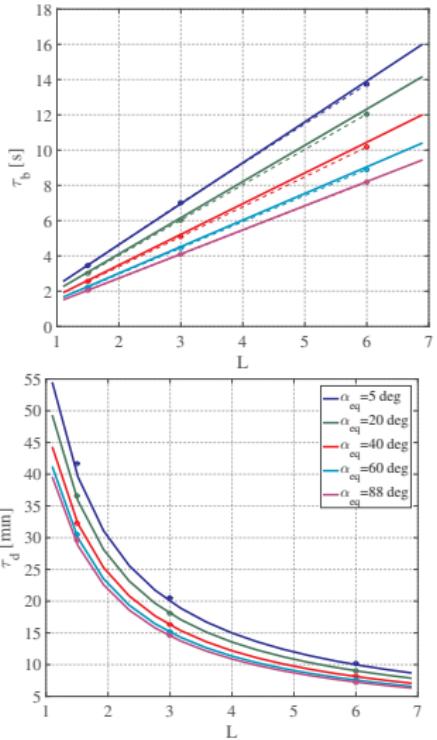
- Bounce period estimate $\tilde{\tau}_b$

$$\text{Latitude}(t) = A_1 \sin \left(2\pi \frac{t}{\tilde{\tau}_b} + \phi_1 \right)$$

- Drift period estimate $\tilde{\tau}_d$

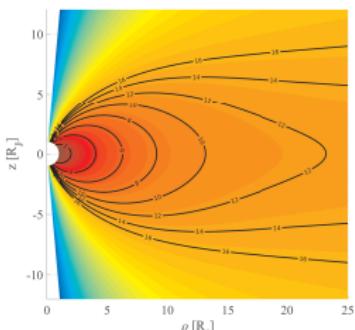
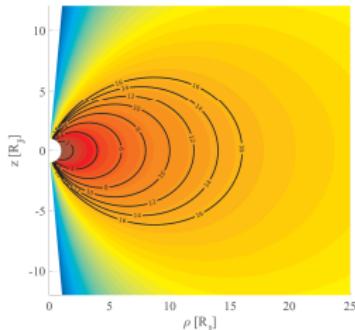
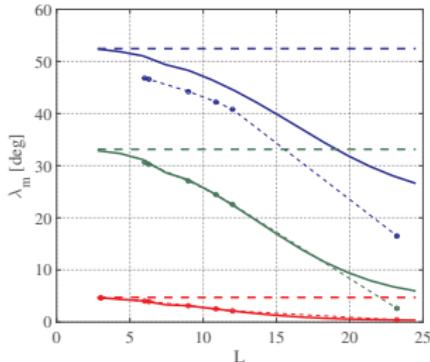
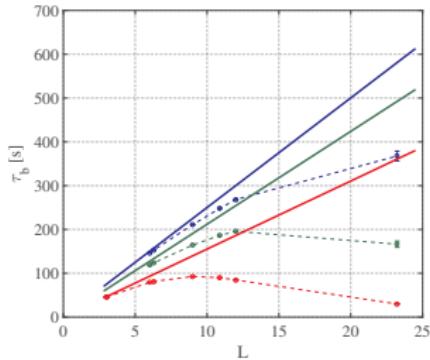
$$\text{Longitude}(t) = \tilde{\tau}_d \left[t + A_2 \sin \left(2\pi \frac{2t}{\tilde{\tau}_b} + \phi_2 \right) \right]$$

Particle Tracing: 1 MeV Proton in Earth Dipole



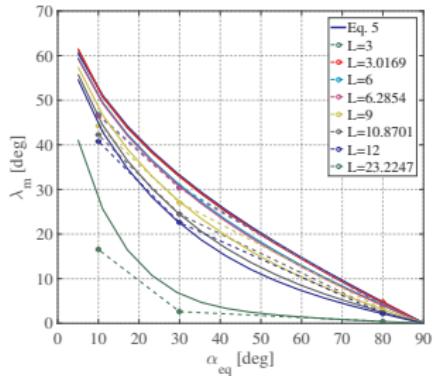
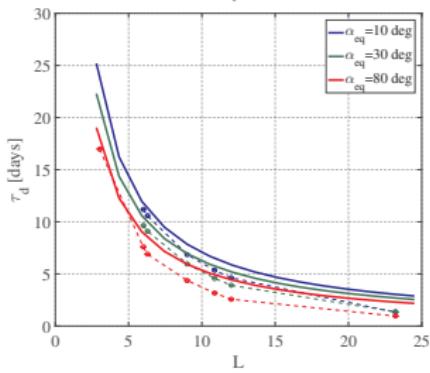
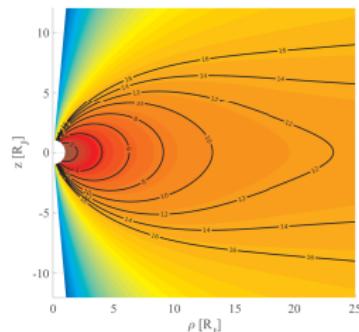
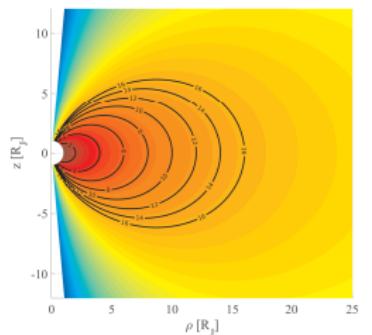
Guio et al. [2016]

1 MeV Proton in Jupiter Magnetodisc λ_m and τ_b



Guio et al. [2016]

1 MeV Proton in Jupiter Magnetodisc τ_d



Guio et al. [2016]

Bounce and Drift Period Integrals: General Formulation

- Mirror point latitude $\lambda_m(R_{\text{eq}}, \alpha_{\text{eq}})$

$$\sin^2(\alpha_{\text{eq}}) = \frac{B(R_{\text{eq}}, 0)}{B_m(r_m, \lambda_m)}$$

- Bounce period $\tau_b (R_{\text{eq}} \equiv LR_P)$

$$\tau_b = LR_P \frac{2(2m)^{1/2}}{W^{1/2}} \Phi(R_{\text{eq}}, \alpha_{\text{eq}})$$

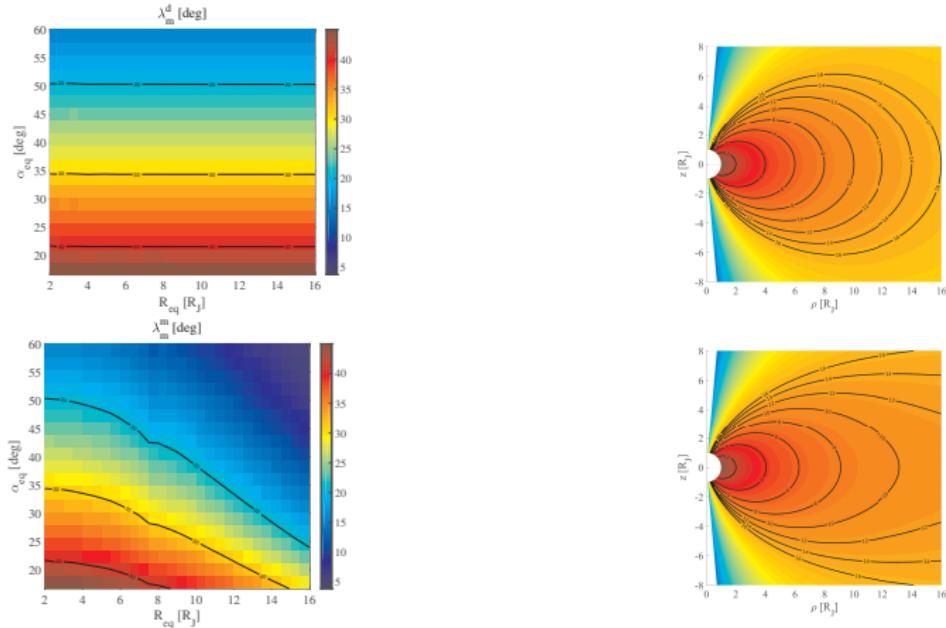
Φ still dimensionless function of geometry but $(R_{\text{eq}}, \alpha_{\text{eq}})$

- Drift period τ_d

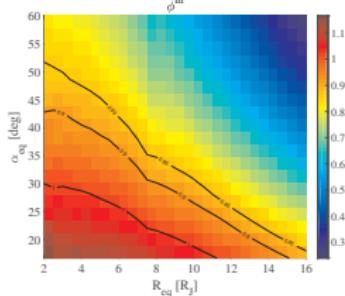
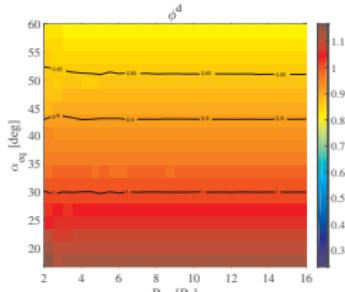
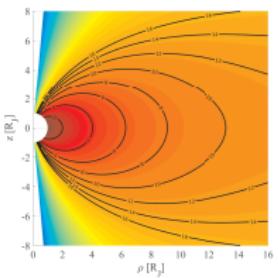
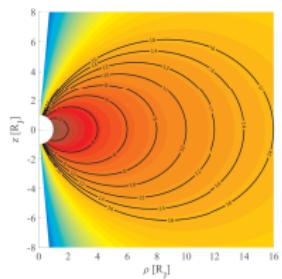
$$\tau_d = \frac{\pi q B_P R_P^2}{3LW} \frac{\Phi(R_{\text{eq}}, \alpha_{\text{eq}})}{\Omega(R_{\text{eq}}, \alpha_{\text{eq}})}, \quad \text{with} \quad \Omega = \Omega_c + \Omega_g$$

Ω 's still dimensionless functions of geometry but $(R_{\text{eq}}, \alpha_{\text{eq}})$

λ_m for Jupiter Magnetodisc [Guio et al., 2017]



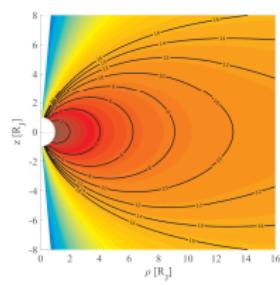
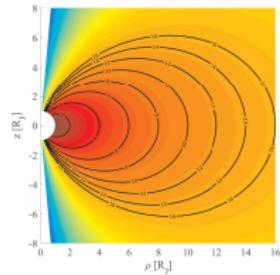
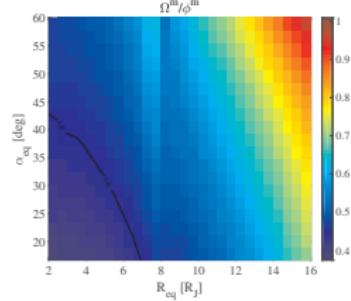
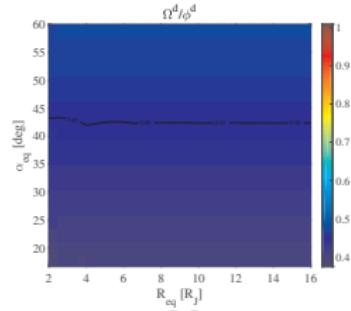
Φ and τ_b for Jupiter Magnetodisc [Guio et al., 2017]



$$\tau_b^d \sim LR_P \frac{2(2m)^{1/2}}{W^{1/2}} (1.30 - 0.58 \sin \alpha_{\text{eq}}),$$

$$\tau_b^m \sim LR_J \frac{2(2m)^{1/2}}{W^{1/2}} (1.27 - 0.37 \sin \alpha_{\text{eq}} - 0.05L \sin \alpha_{\text{eq}}),$$

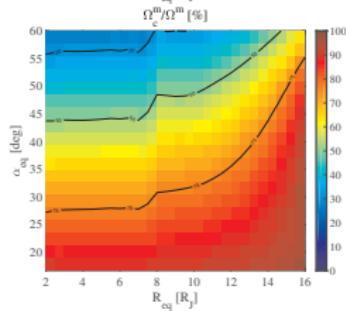
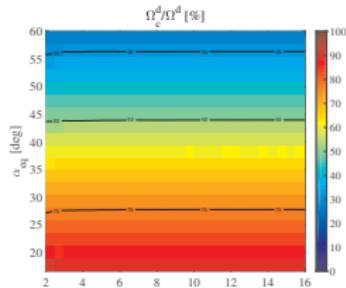
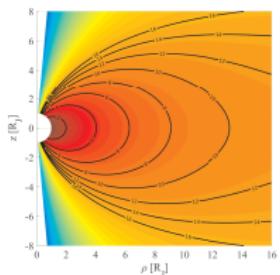
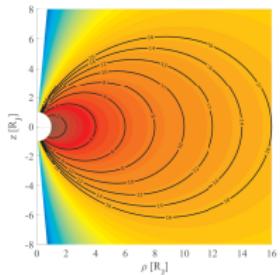
Ω/Φ and τ_d for Jupiter Magnetodisc



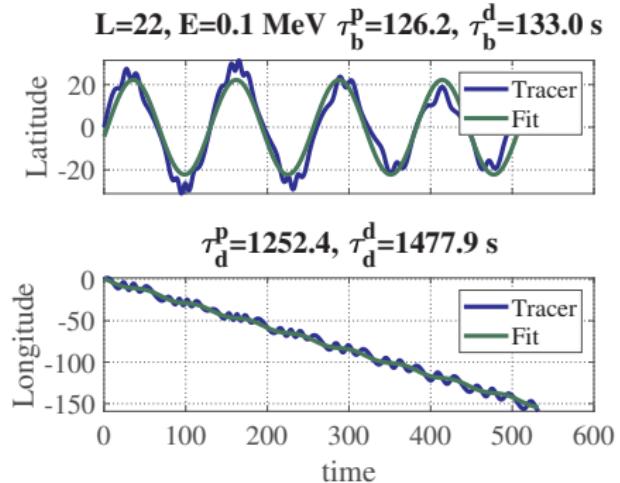
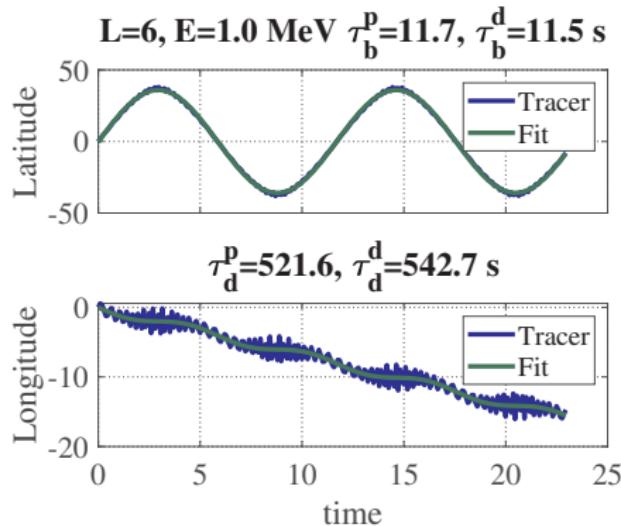
$$\tau_d^d \sim \frac{\pi q B_P R_P^2}{3LW} \frac{1}{0.35 + 0.15 \sin \alpha_{\text{eq}}}$$

$$\tau_d^m \sim \frac{\pi q B_J R_J^2}{3LW} \frac{1}{0.40 - 0.06 \sin \alpha_{\text{eq}} + 0.04L \sin \alpha_{\text{eq}}}$$

Ω_c/Ω for Jupiter Magnetodisc



Particle Tracing: Guiding Centre Approximation Breaking



- $r_g \gg |B/\nabla B|$ (energy, R_{eq} and/or α_{eq} increase)
- Check this out yourself in the practicals this afternoon!

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