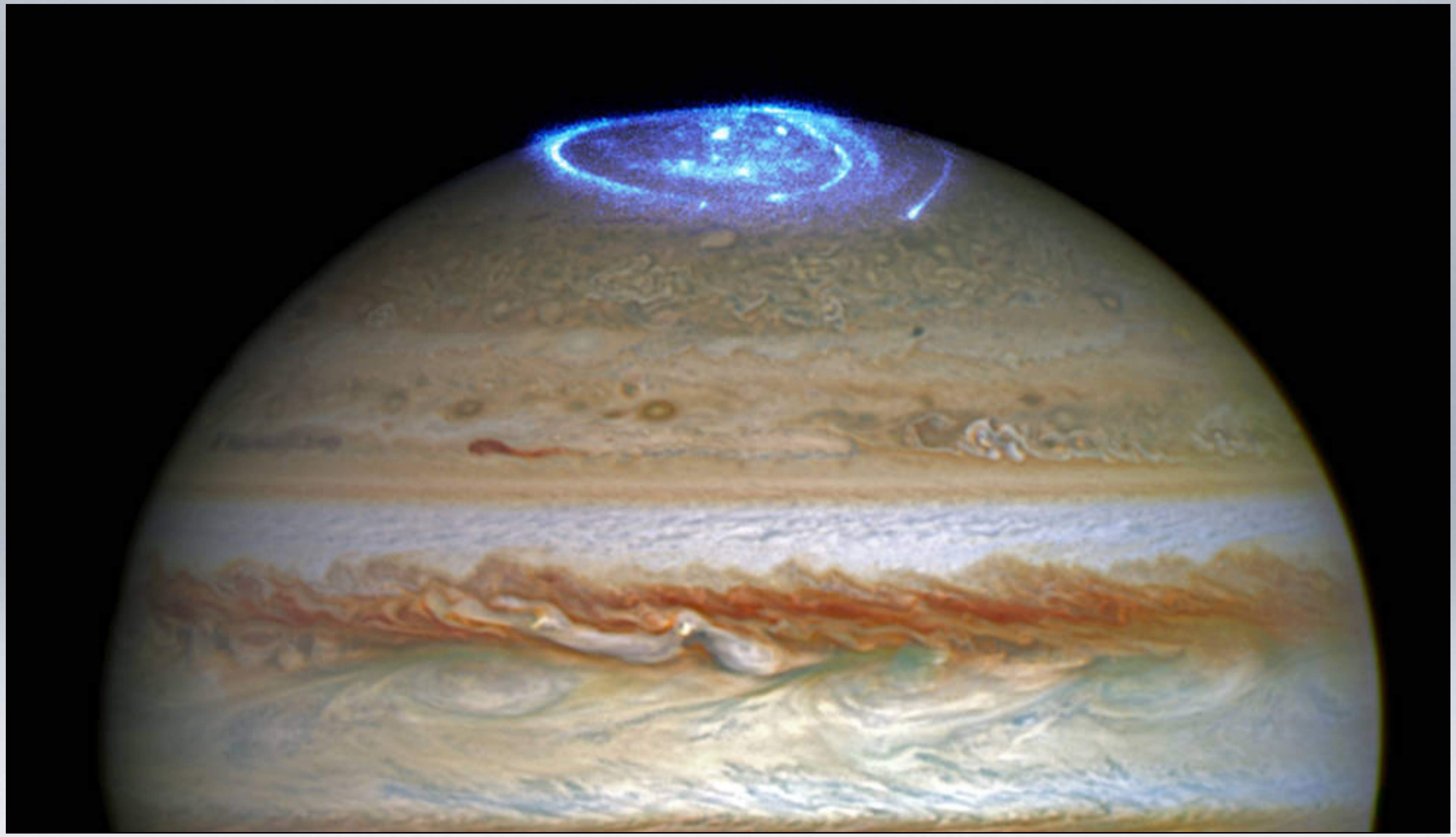
A short introduction to…
Particle motion in electromagnetic fields



J. D. Nichols, S.W.H. Cowley

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Typical interplanetary parameters at 1 au

Number density	10 cm^{-3}
α particles	5%
Flow speed (radial)	450 km s^{-1}
Temperature	10^5 K (~10 eV)
Magnetic Field	10 nT
Dynamic Pressure	3 nPa

Mean free path ~ 1 au

Hence, we can study charged particle motion in the *absence of collisions*

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Equation of motion

The basic equation of motion is:

Particle mass Particle charge Particle velocity

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Time Electric field Magnetic field

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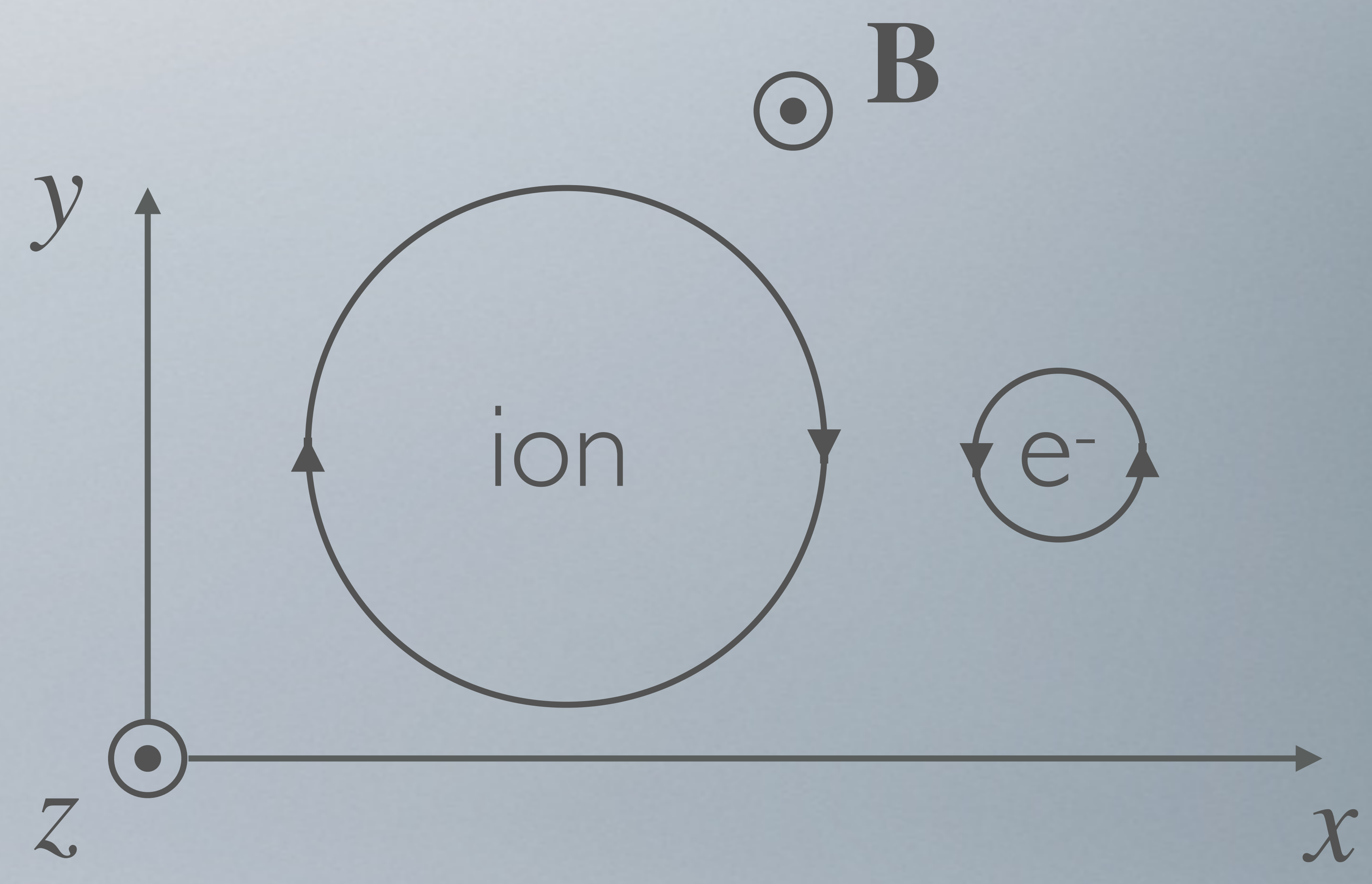
Motion in a uniform steady **B** (zero **E**)

If we take $\mathbf{B} = B\,\underline{\hat{\mathbf{z}}}$ and $q = e$

$$\frac{d\mathbf{v}}{dt} = \frac{e}{m}\mathbf{v} \times \mathbf{B} = \frac{e}{m}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix}$$



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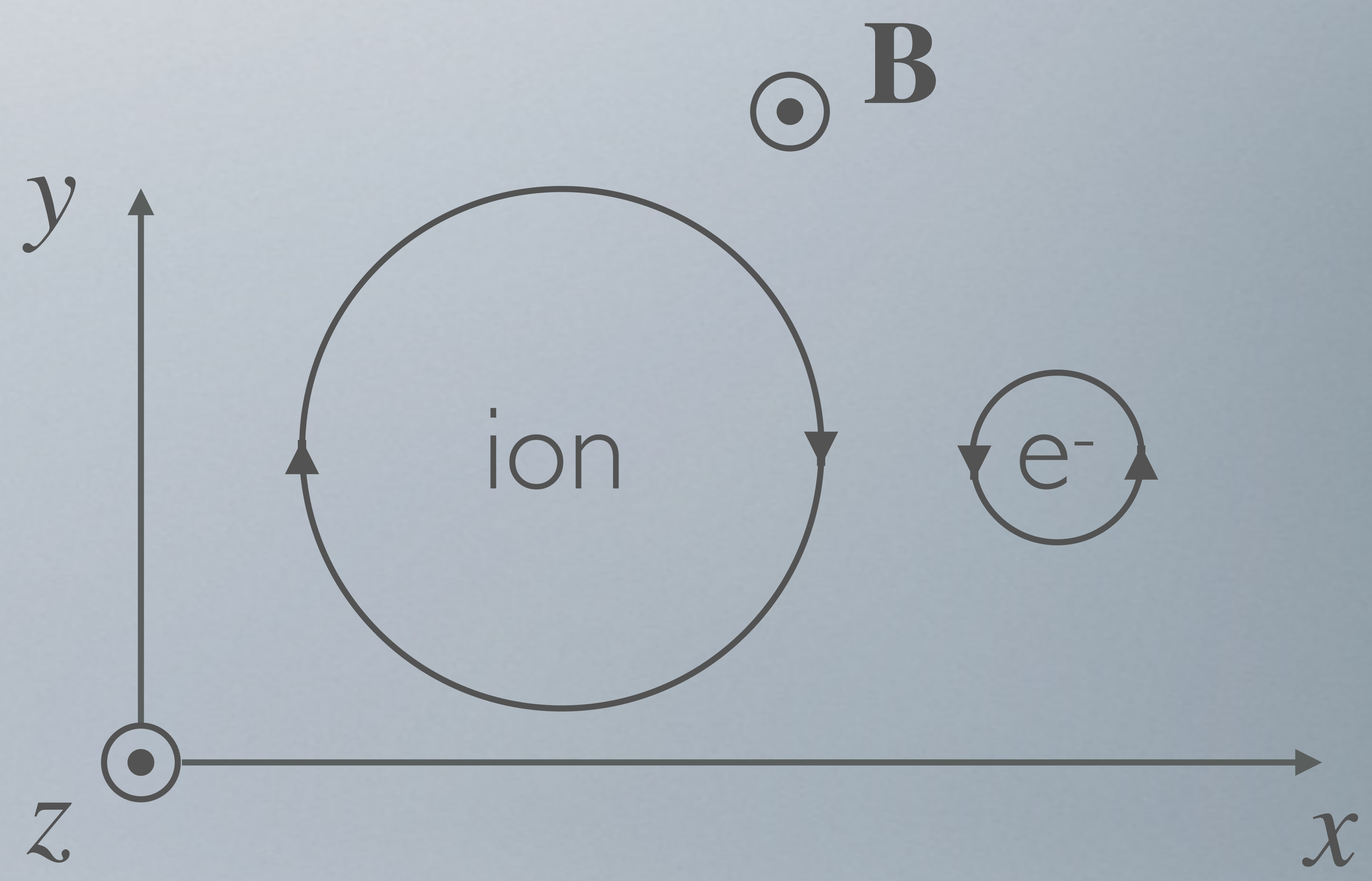

$$\frac{dv_x}{dt} = \frac{e}{m}v_y B_z \qquad \frac{dv_y}{dt} = -\frac{e}{m}v_x B_z$$

Circular motion perpendicular to **B**

$$\frac{dv_z}{dt} = 0$$

Uniform motion parallel to **B**

Motion in a uniform steady **B** (zero **E**)

$$\frac{dv_x}{dt} = \frac{e}{m} v_y B_z \quad (1)$$

$$\frac{dv_y}{dt} = \frac{e}{m} v_x B_z \quad (2)$$

Take equation (1) and substitute into equation (2):

$$\frac{d^2 v_x}{dt^2} = \left(\frac{eB}{m}\right) \frac{dv_y}{dt} = -\left(\frac{eB}{m}\right)^2 v_x$$

This is the equation for SHM for v_x with angular frequency

$$\Omega = \left(\frac{eB}{m}\right)$$

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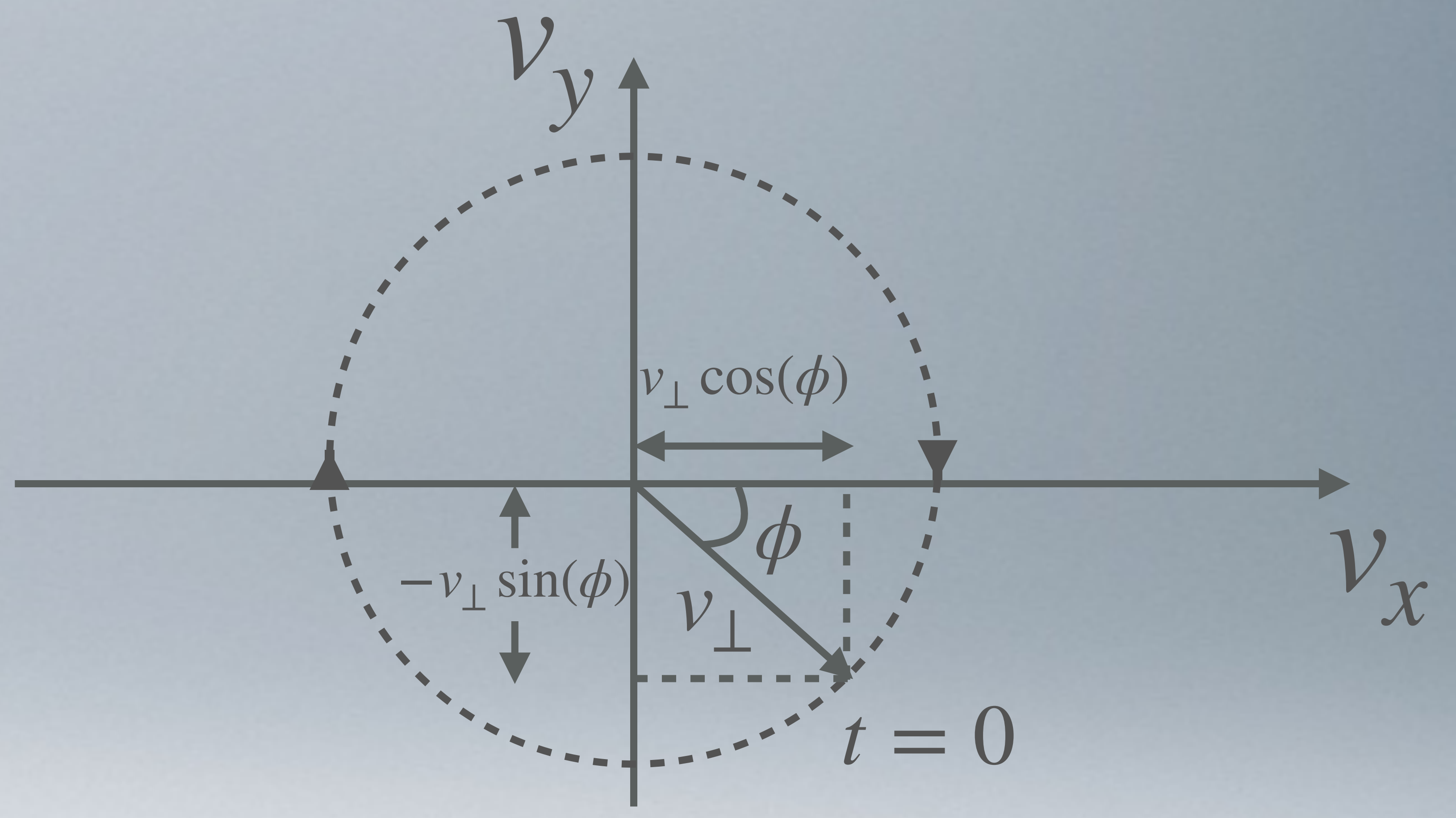
Motion in a uniform steady **B** (zero **E**)

The general solution is: $\boxed{v_x = v_\perp \cos(\Omega t + \phi)}$

Phase (constant)

Amplitude (constant speed ⊥ to **B**)

Substitute into (1) for v_y: $\boxed{v_y = -v_\perp \sin(\Omega t + \phi)}$



Velocity vector rotates with constant speed $v_\perp = \sqrt{v_x^2 + V_y^2}$ with angular frequency $\boldsymbol{\Omega}$

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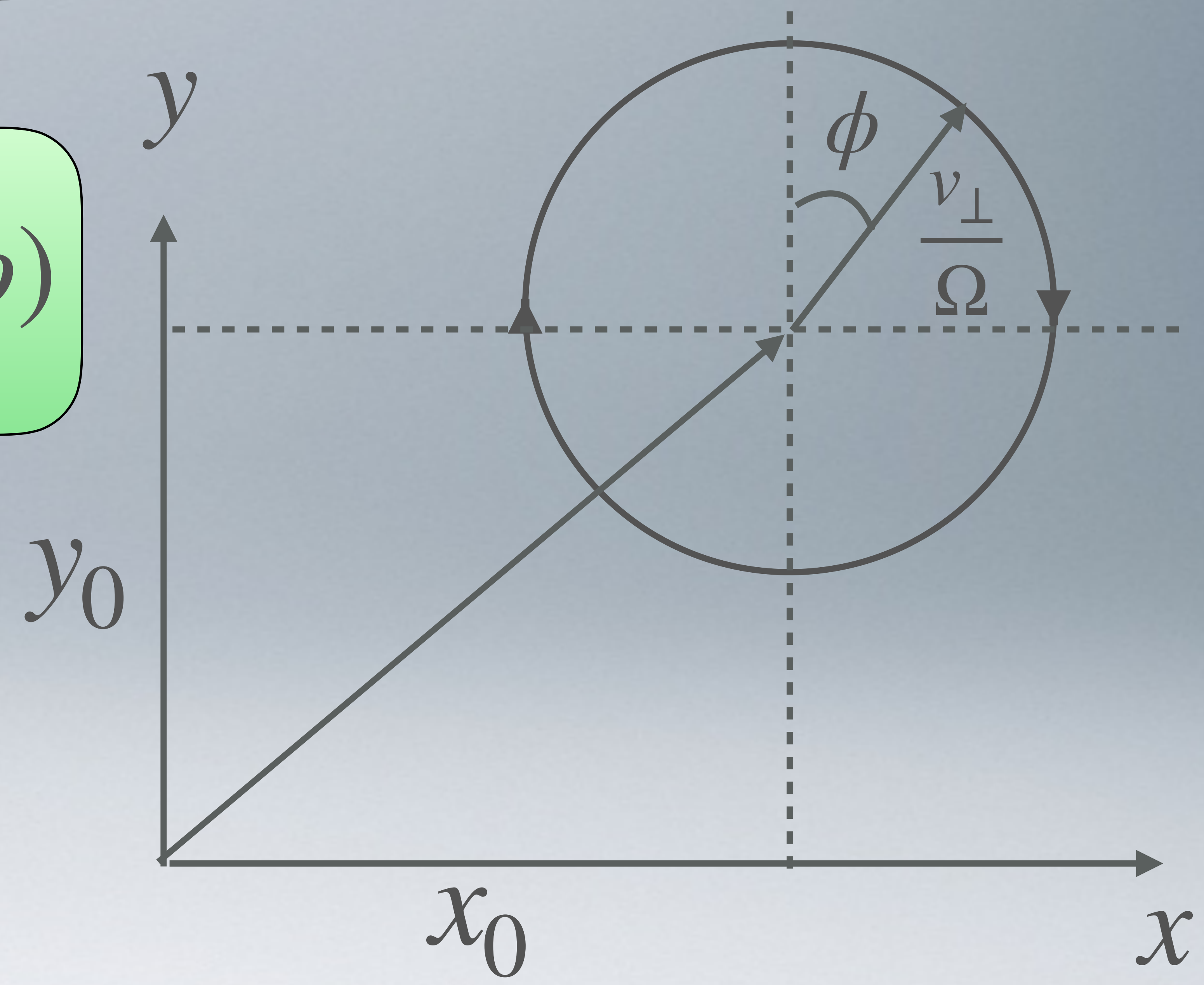
Motion in a uniform steady **B** (zero **E**)

Integrating again:

$$\boxed{x(t) = x_0 + \frac{v_\perp}{\Omega}\sin(\Omega t + \phi)}$$

$$\boxed{y(t) = y_0 + \frac{v_\perp}{\Omega}\cos(\Omega t + \phi)}$$

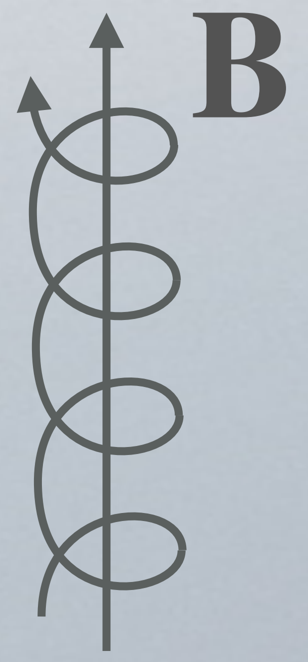
Integration constant



This corresponds to motion in a circle about centre (x_0, y_0) with radius $(v_\perp/\Omega)$ and angular frequency Ω

Motion in a uniform steady **B** (zero **E**)

The overall motion is a *helix*



All particles of the same species rotate at the same angular frequency, $\Omega = \left(\frac{eB}{m}\right)$, the *gyrofrequency*, regardless of their speed. Ions (positively charged) and electrons (negatively charged) rotate in opposite senses.

The circle radius is given by $r_g = \frac{v_\perp}{\Omega} = \frac{mv_\perp}{eB}$, termed the *gyroradius*

Motion in a uniform steady **B** (zero **E**)

Quick proof:

Given the motion is a circle, the previous results follow quickly:

$$m\frac{d\mathbf{v}}{dt} = -\frac{mv_\perp^2}{r_g}\hat{\mathbf{r}} = -ev_\perp B\hat{\mathbf{r}}$$

$$\Rightarrow -\frac{mv_\perp^2}{r_g} = -ev_\perp B$$

$$\boxed{\Rightarrow r_g = \frac{mv_\perp}{eB}}$$ and $$\boxed{\Omega = \frac{v_\perp}{r_g} = \frac{eB}{m}}$$

gyroradius gyrofrequency

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Motion in a non-uniform steady **B** (zero **E**)

Constancy of particle speed:

Equation of motion: $m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{v}\times\mathbf{B}\right)$

Kinetic energy

Dot product with $\mathbf{v}$: $m\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = m\frac{d}{dt}\left(\frac{\mathbf{v}\cdot\mathbf{v}}{2}\right) = \frac{d}{dt}\left(\frac{mv^2}{2}\right)$

$= q\mathbf{v}\cdot\left(\mathbf{v}\times\mathbf{B}\right)$

$= 0$ since $\mathbf{v}\times\mathbf{B}$ is perpendicular to $\mathbf{v}$

Hence, $\boxed{\mathrm{KE} = \frac{mv^2}{2} = \mathrm{const}}$ or, equivalently, $\boxed{|\mathbf{v}| = \mathrm{const}}$

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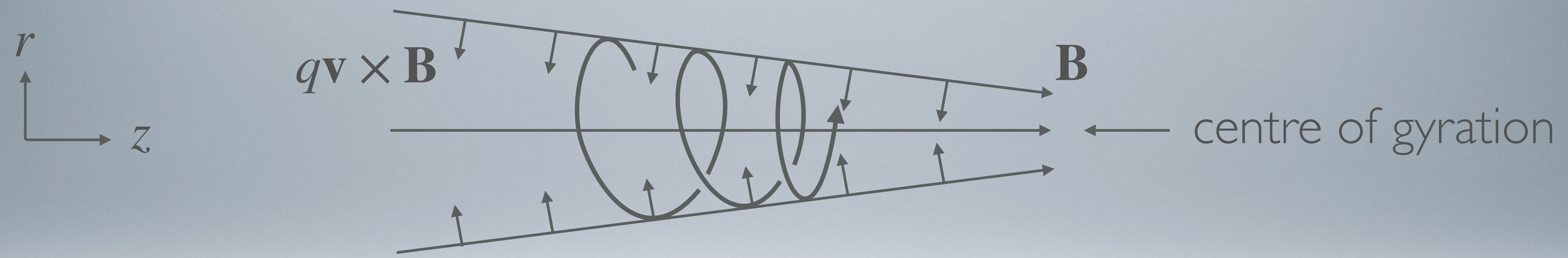
Motion in a non-uniform steady **B** (zero **E**)

Change in magnetic field strength along **B**



The Lorentz force is perpendicular to **B** and hence has a component pointing away from the direction of increasing field strength. This decreases $v_{\|}$ and since $|\mathbf{v}|$ is constant, $v_{\perp}$ must increase. Eventually, $v_{\|} \rightarrow \mathbf{0}$ and the particle is repelled ("mirrored") from the region of increasing **B**.

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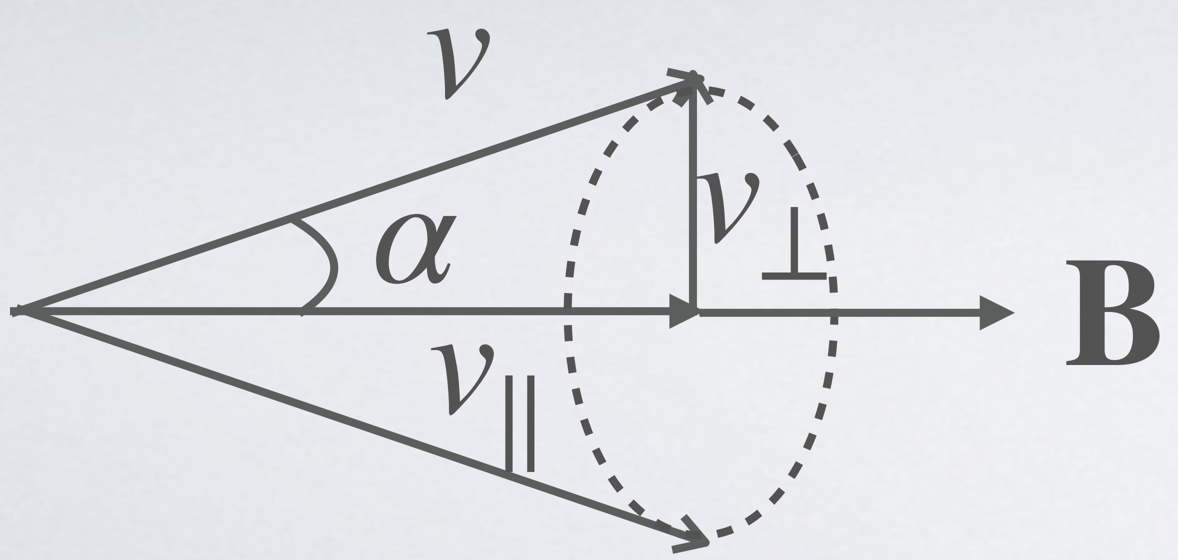
Motion in a non-uniform steady **B** (zero **E**)

It can be shown that if the field strength varies slowly (i.e. $B_r \ll B_z$) then the following holds:

$$\boxed{\frac{v_\perp^2}{B} = \text{const}}$$

The "first adiabatic invariant"

which, when introducing the particle's 'pitch angle' α, results in



$$\boxed{\frac{\sin^2 \alpha}{B} = \text{const}}$$

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Motion in a non-uniform steady **B** (zero **E**)

A particle mirrors when $v_{||} = 0$, i.e. when $\alpha = 90°$ or $\sin\alpha = 1$. Hence if a particle has pitch angle α at a location with field strength B, it mirrors at field strength B_m given by

$$\boxed{B_m = \frac{B}{\sin^2\alpha}}$$

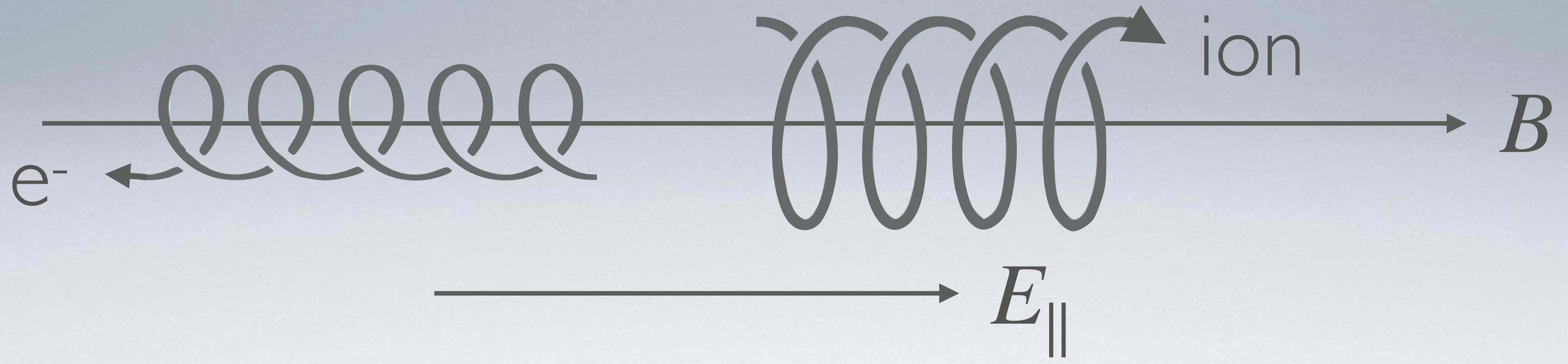
The mirror point depends <u>only on pitch angle</u> - not particle energy or species
Particles with smaller pitch angle mirror at higher field strengths
The magnetic flux threading each gyroradius is constant
Particles can be trapped on e.g. dipole fields or magnetic bottles

Effect of $\mathbf{E}$ parallel to $\mathbf{B}$

If we take $\mathbf{B} = B\,\underline{\hat{\mathbf{z}}}$ as before and $\mathbf{E} = E_{\|}\,\underline{\hat{\mathbf{z}}}$, the equation of motion is

$$\boxed{\frac{dv_z}{dt} = \frac{q}{m}E_{\|}} \quad \Rightarrow v_z = v_{z0} + \frac{qE_{\|}}{m}t$$

Hence, electrons travel anti-parallel to $\mathbf{E}$, ions parallel to $\mathbf{E}$, i.e. the charge separates

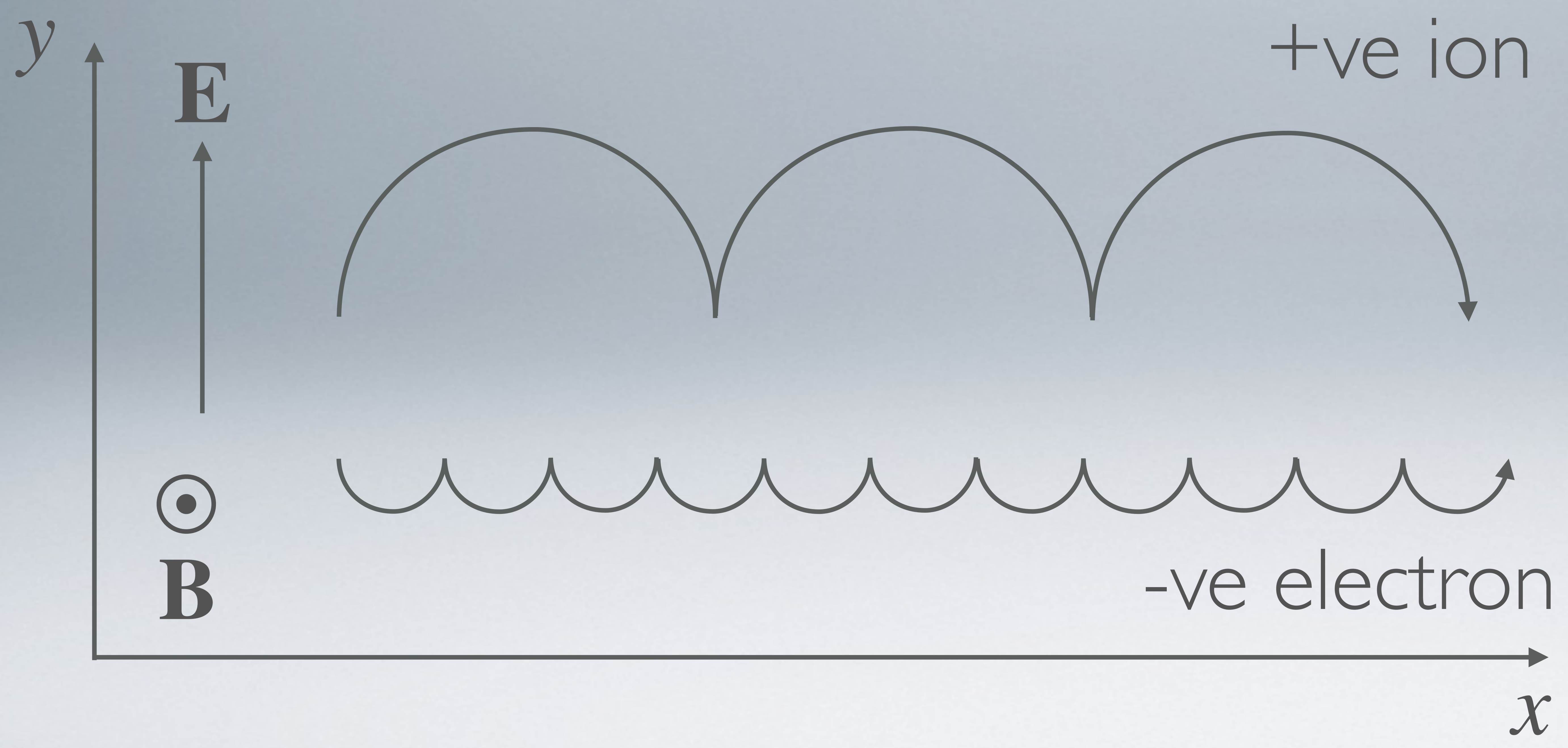


This sets up an E field opposing the original $E_{\|}$, which quickly drops to zero

Hence, usually (not always) $\boxed{E_{\|} = 0}$

Effect of $\mathbf{E}$ perpendicular to $\mathbf{B}$

If we take $\mathbf{B} = B\,\hat{\underline{\mathbf{z}}}$ as before and $\mathbf{E} = E_\perp\,\hat{\underline{\mathbf{y}}}$:



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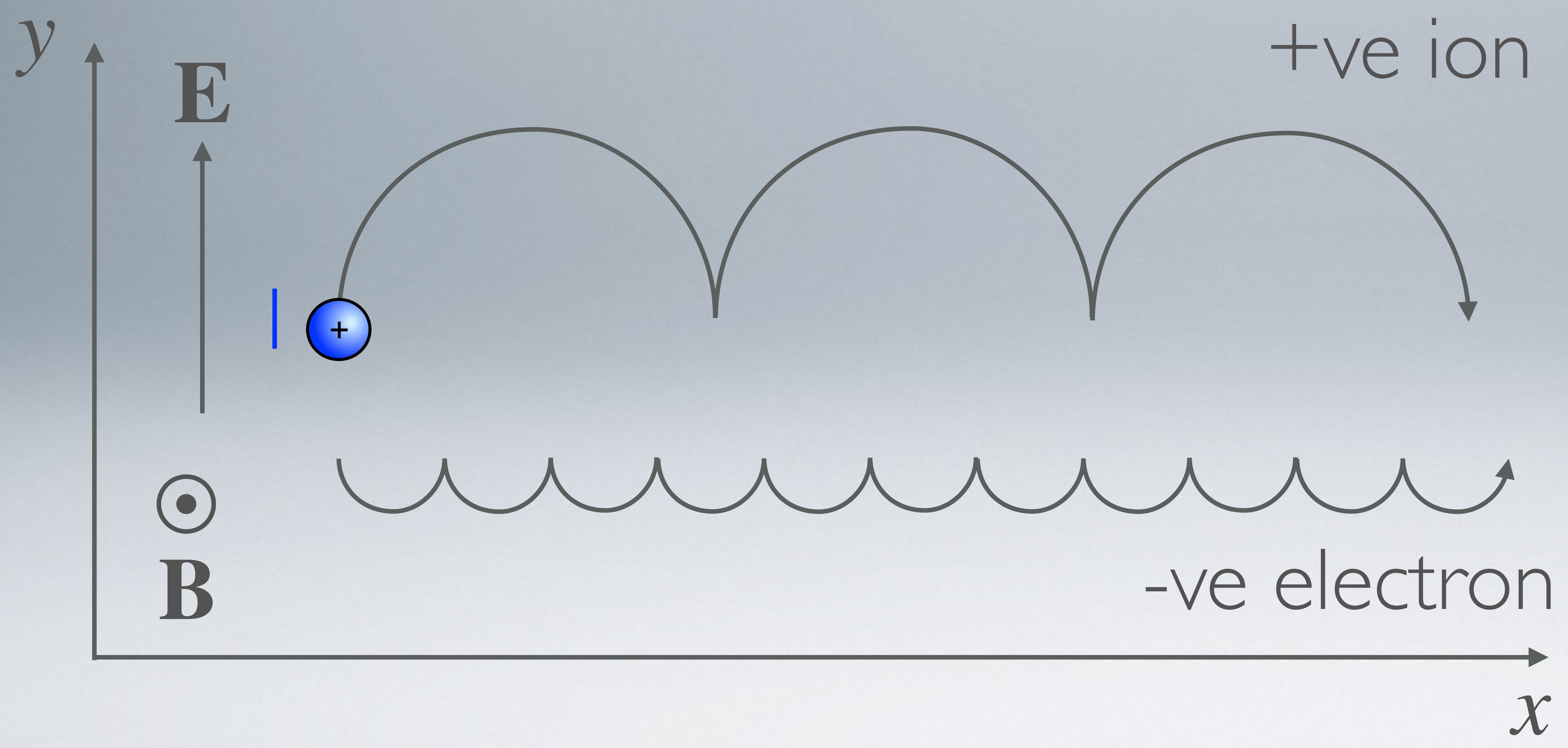

1. Consider a positive particle, initially at rest. It accelerates in the direction of the electric field, ie. $+\hat{\mathbf{y}}$.

Effect of $\mathbf{E}$ perpendicular to $\mathbf{B}$

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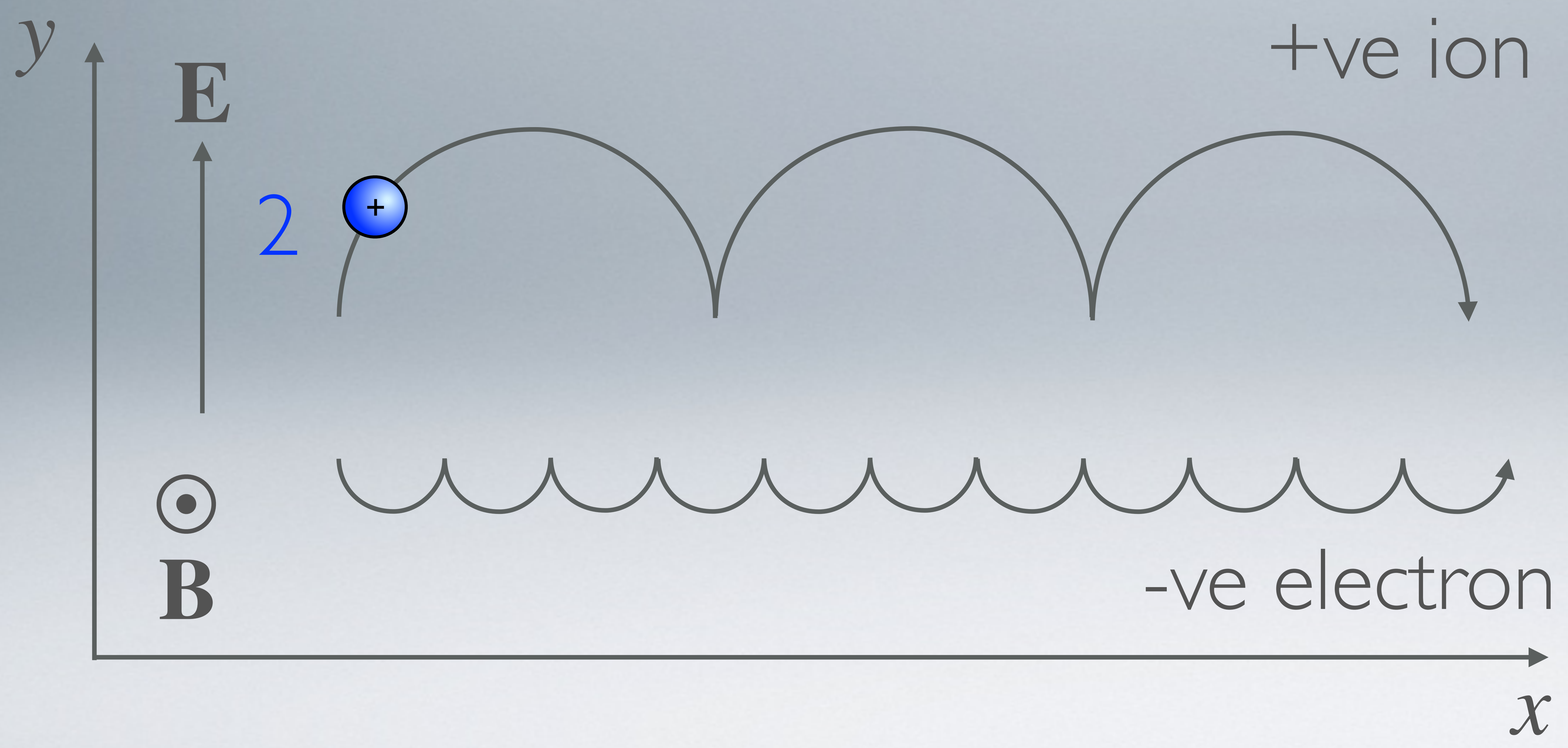
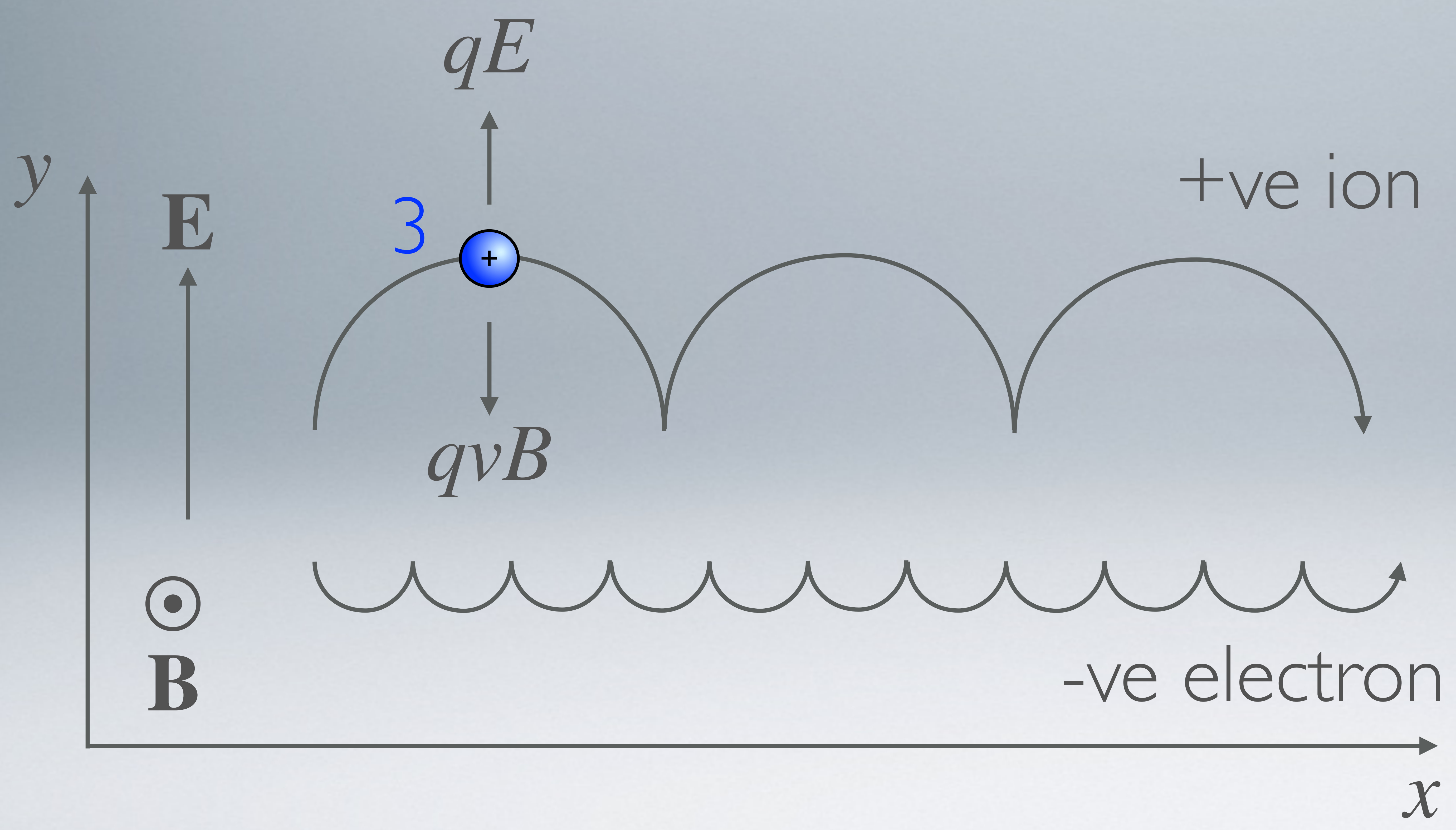


2. It begins to turn in the $+\hat{\mathbf{x}}$ direction due to $\mathbf{v} \times \mathbf{B}$

Effect of $\mathbf{E}$ perpendicular to $\mathbf{B}$

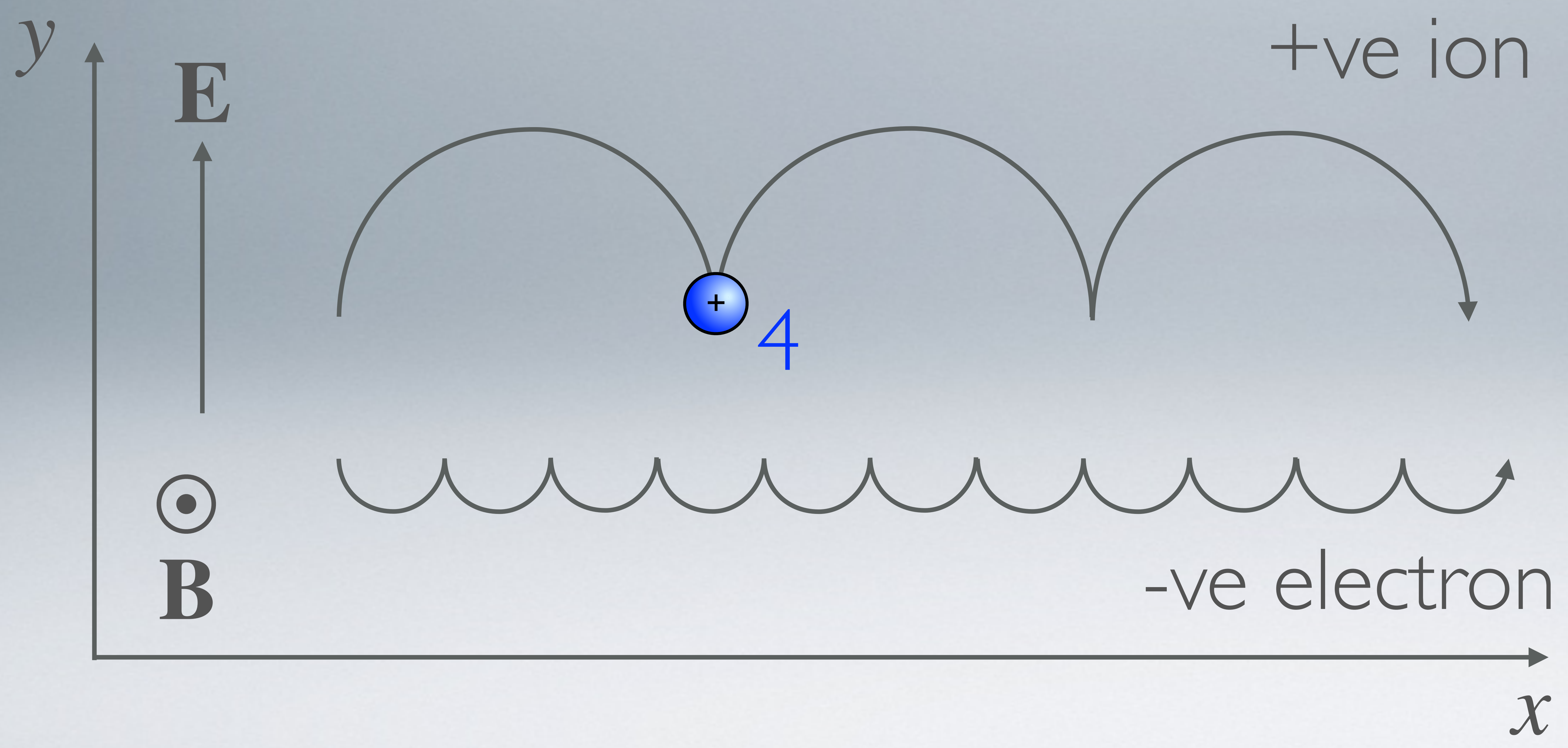
If we take $\mathbf{B} = B\,\underline{\hat{\mathbf{z}}}$ as before and $\mathbf{E} = E_\perp\,\underline{\hat{\mathbf{y}}}$:



3. When $v_y = 0$ the $q\mathbf{v} \times \mathbf{B}$ force is opposite to the $q\mathbf{E}$ force but twice as strong, so turning continues

Effect of $\mathbf{E}$ perpendicular to $\mathbf{B}$

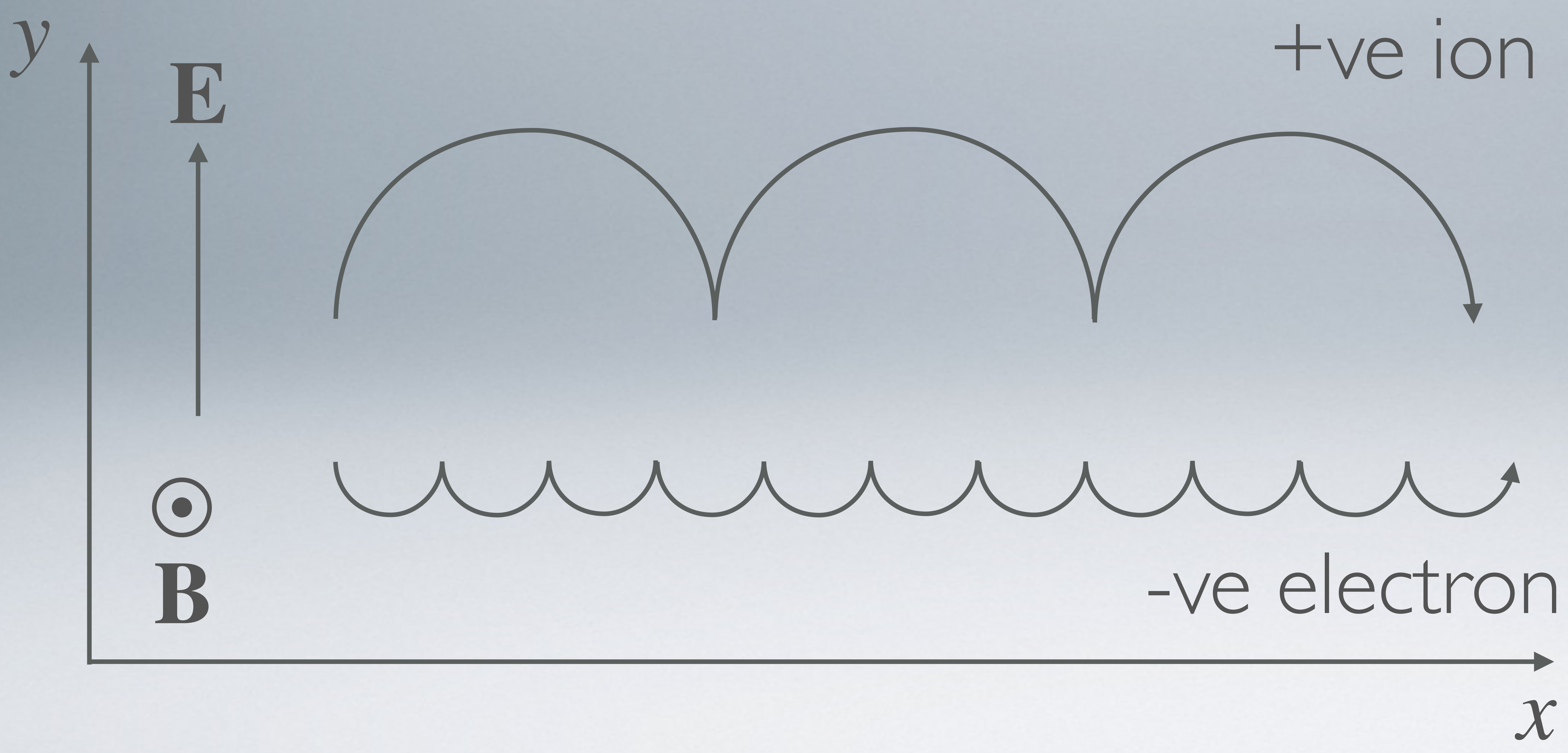
If we take $\mathbf{B} = B\,\hat{\underline{\mathbf{z}}}$ as before and $\mathbf{E} = E_\perp\,\hat{\underline{\mathbf{y}}}$:



4. Until it comes to rest and the cycle repeats

Effect of $\mathbf{E}$ perpendicular to $\mathbf{B}$

If we take $\mathbf{B} = B\,\underline{\hat{\mathbf{z}}}$ as before and $\mathbf{E} = E_\perp\,\underline{\hat{\mathbf{y}}}$:



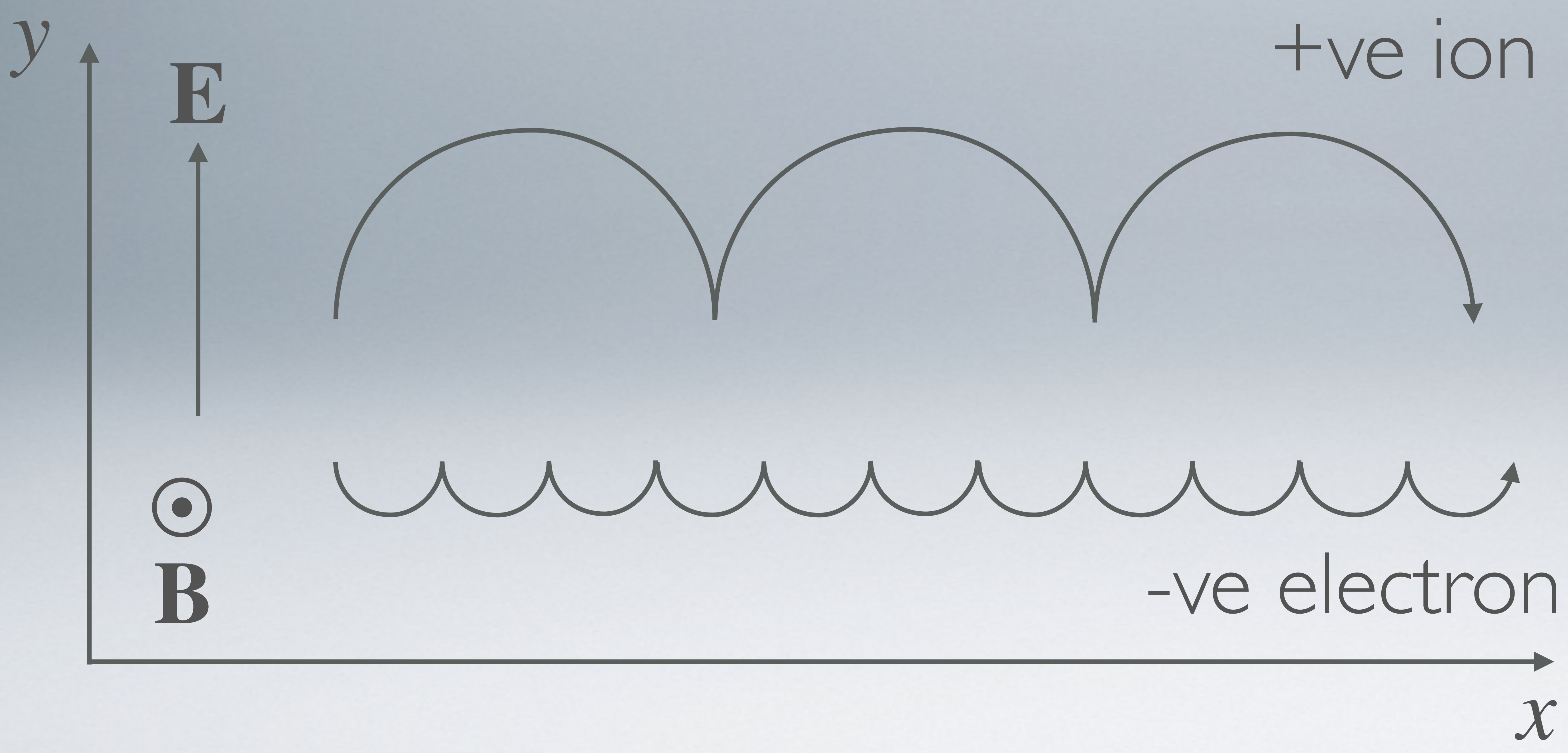
All particles drift at the same speed in the $+\hat{\mathbf{x}}$ direction, <u>independent of charge and mass</u>. The drift velocity is

$$\boxed{\mathbf{v}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2}}$$

and so the drift speed is $\boxed{\left(\frac{E}{B}\right)}$

Effect of $\mathbf{E}$ perpendicular to $\mathbf{B}$

If we take $\mathbf{B} = B\,\underline{\hat{\mathbf{z}}}$ as before and $\mathbf{E} = E_\perp\,\underline{\hat{\mathbf{y}}}$:

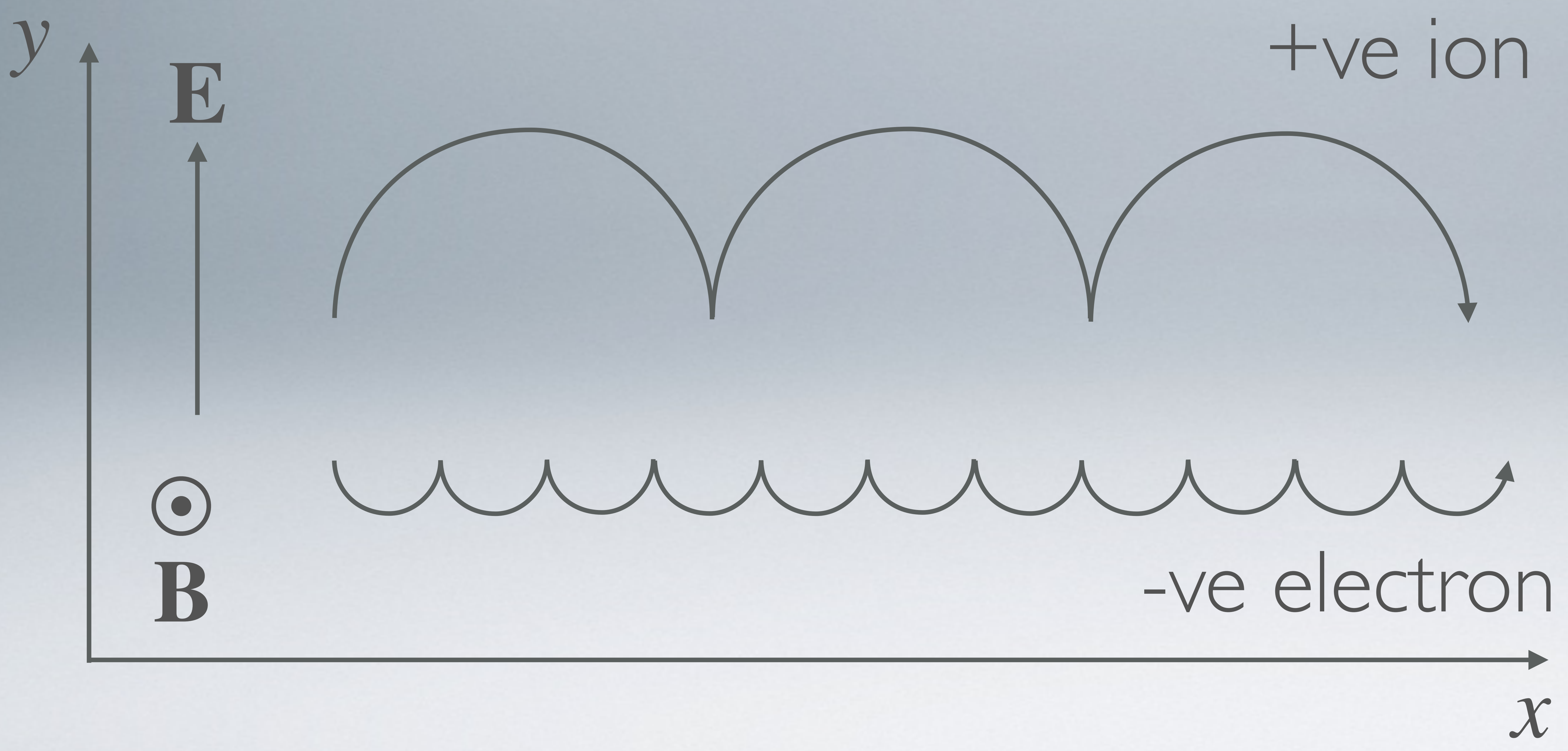


Note that if we choose a frame of reference moving at speed (E/B) in the $+\hat{\mathbf{x}}$ direction, the particle simply appears to gyrate - there is no E field in this frame. Hence, E fields are frame dependent, but B fields are not*

*non-relativistic transformation

Effect of $\mathbf{E}$ perpendicular to $\mathbf{B}$

If we take $\mathbf{B} = B\,\hat{\underline{\mathbf{z}}}$ as before and $\mathbf{E} = E_\perp\,\hat{\underline{\mathbf{y}}}$:

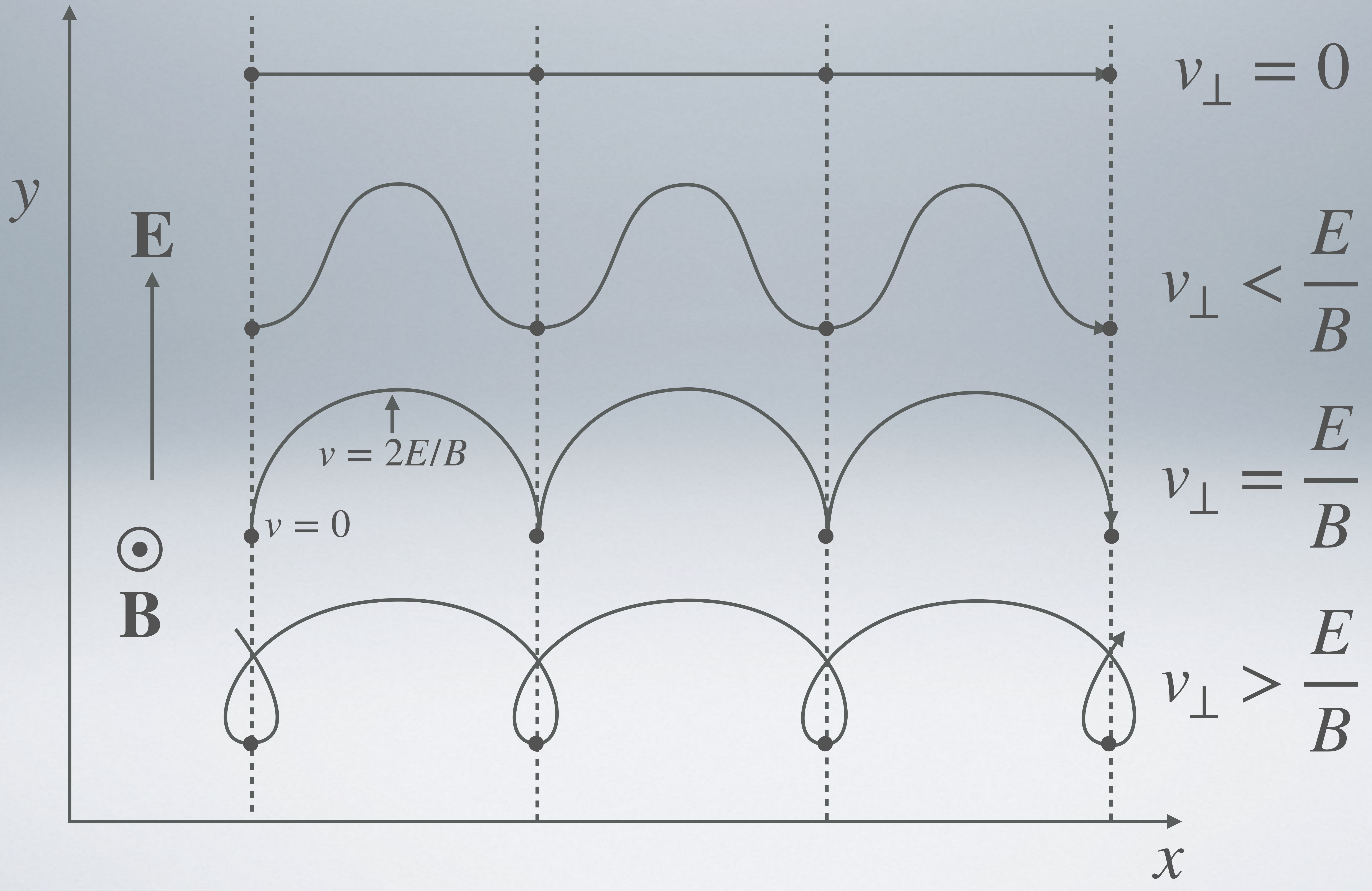


Equivalently, if in a given frame of reference there is a flow velocity $\mathbf{v}$ then there exists an E field in that frame given by

$$\boxed{\mathbf{E} = -\,\mathbf{v} \times \mathbf{B}}$$

Effect of $\mathbf{E}$ perpendicular to $\mathbf{B}$

The overall motion is the sum of gyration and $\mathbf{E} \times \mathbf{B}$ drift:



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Frozen-in Flow

Consider an electromagnetic field $\mathbf{E}(r,t), \mathbf{B}(r,t)$ which varies slowly in space and time compared with the gyroradii and gyroperiods of the particles, respectively.

Assume $\mathbf{E}$ is everywhere perpendicular to $\mathbf{B}$

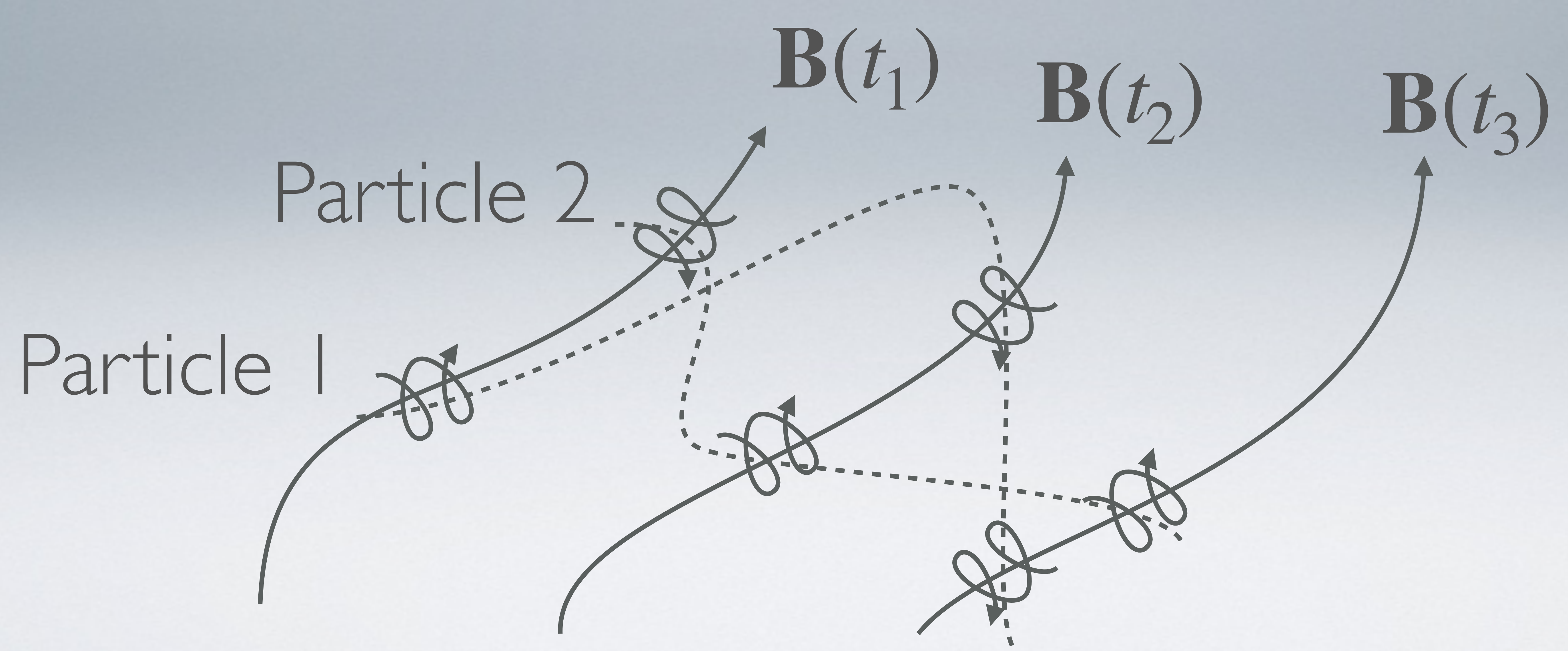
Faraday's law is obeyed, i.e. $\dfrac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$

Particle motion then comprises velocities parallel and perpendicular to the field (gyration) and $\mathbf{E} \times \mathbf{B}$ drift.

Frozen-in Flow

In this case, the motion has a special property:

As the particles move, their guiding centres remain on the same field line for all time



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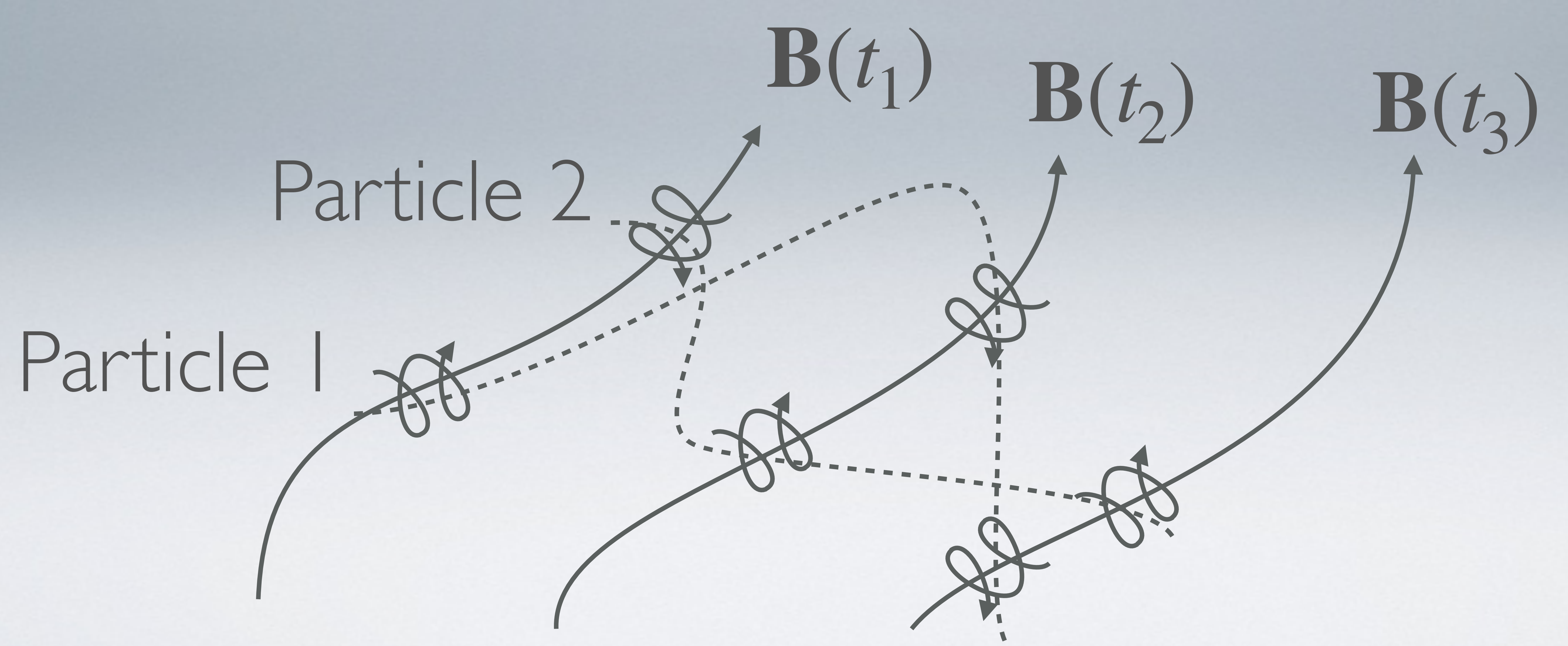
Frozen-in Flow

Either consider the plasma carrying the magnetic field with it
or equivalently
Moving field lines carry the plasma particles

Plasma energy dominant

Magnetic energy dominant



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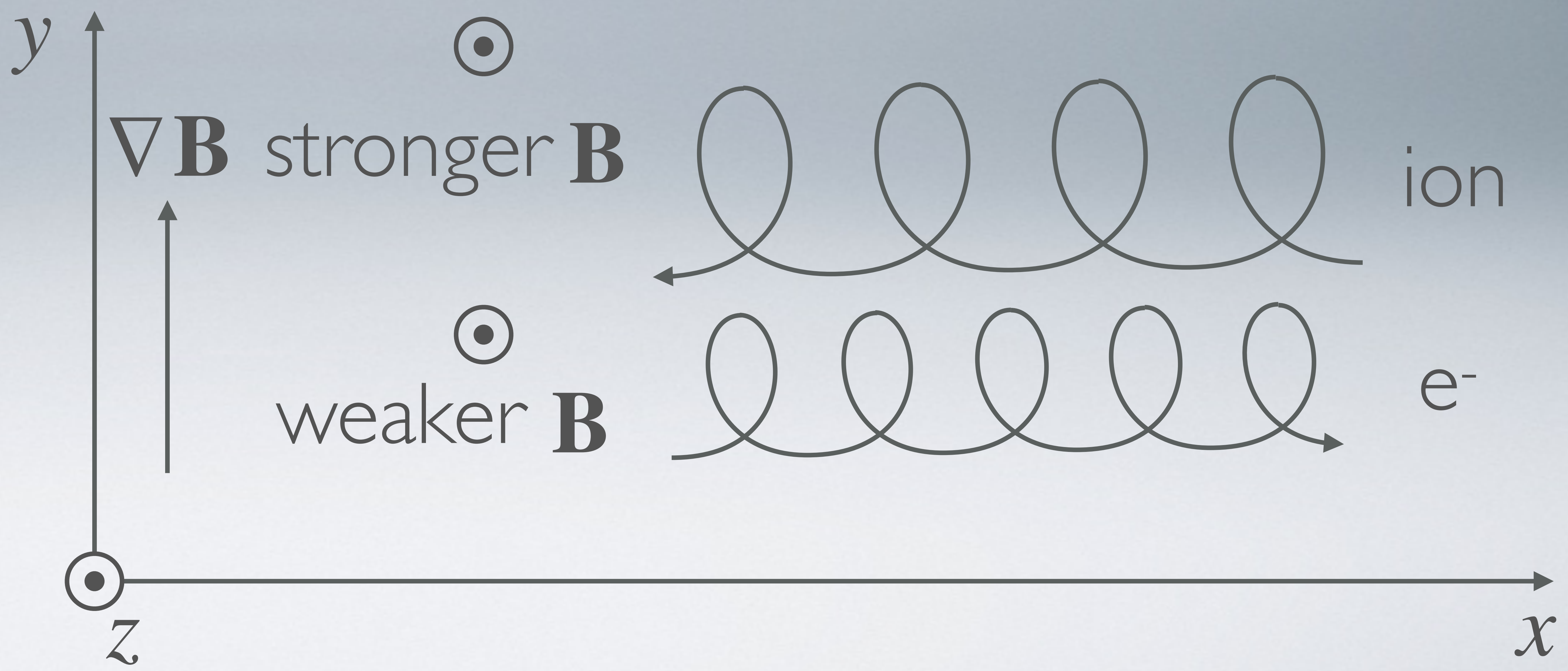
Breakdown of Frozen-in approximation

If the magnetic field strength varies across the field on a scale comparable with the gyroradii of the particles, the particles drift and the frozen in approximation breaks down. This is called the "grad-B drift".

This arises since particle gyroradius is inversely proportional to B such that larger B leads to smaller radius and vice versa



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Breakdown of Frozen-in approximation

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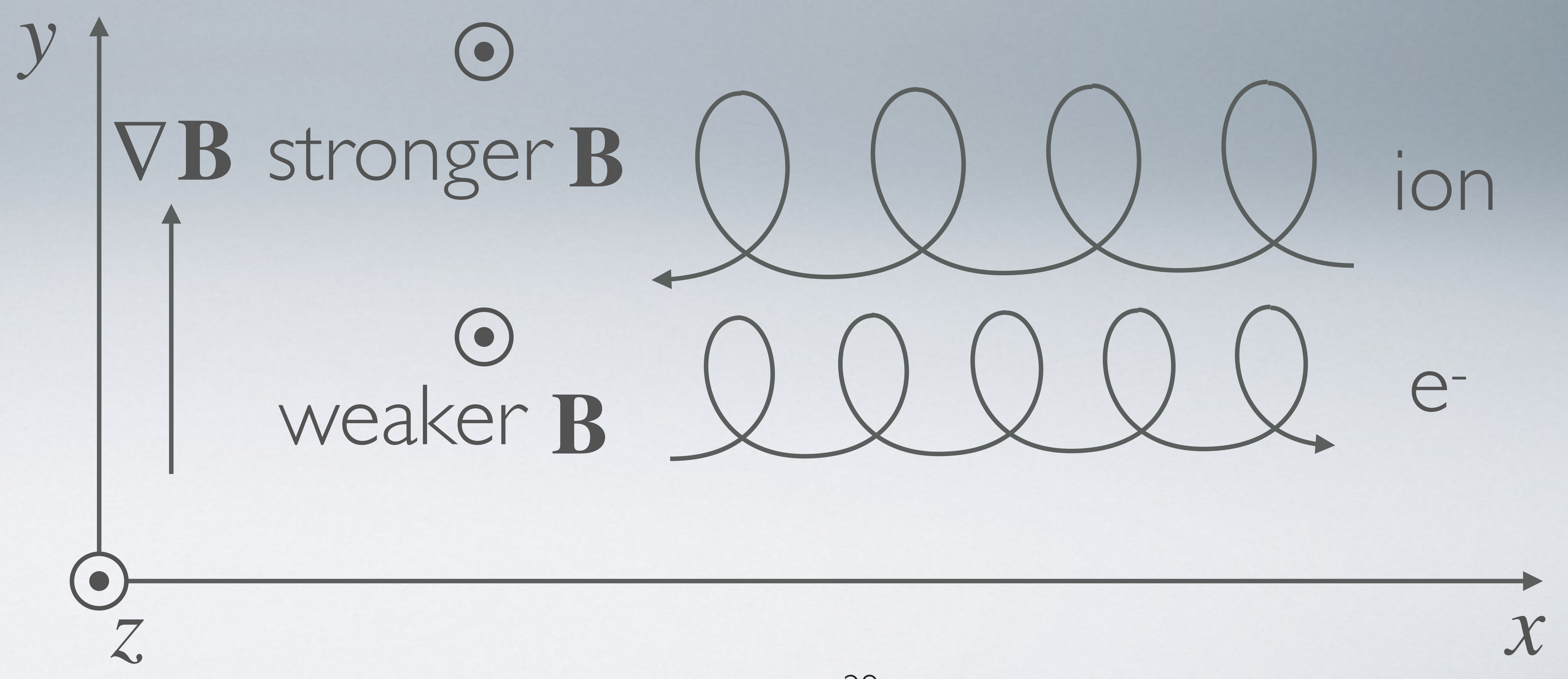
Drift velocity:

$$\boxed{\mathbf{v}_{\nabla B} = \frac{W_\perp}{qB^3}\mathbf{B} \times \nabla \mathbf{B}}$$

Perpendicular energy

$$W_\perp = \frac{mv_\perp^2}{2}$$



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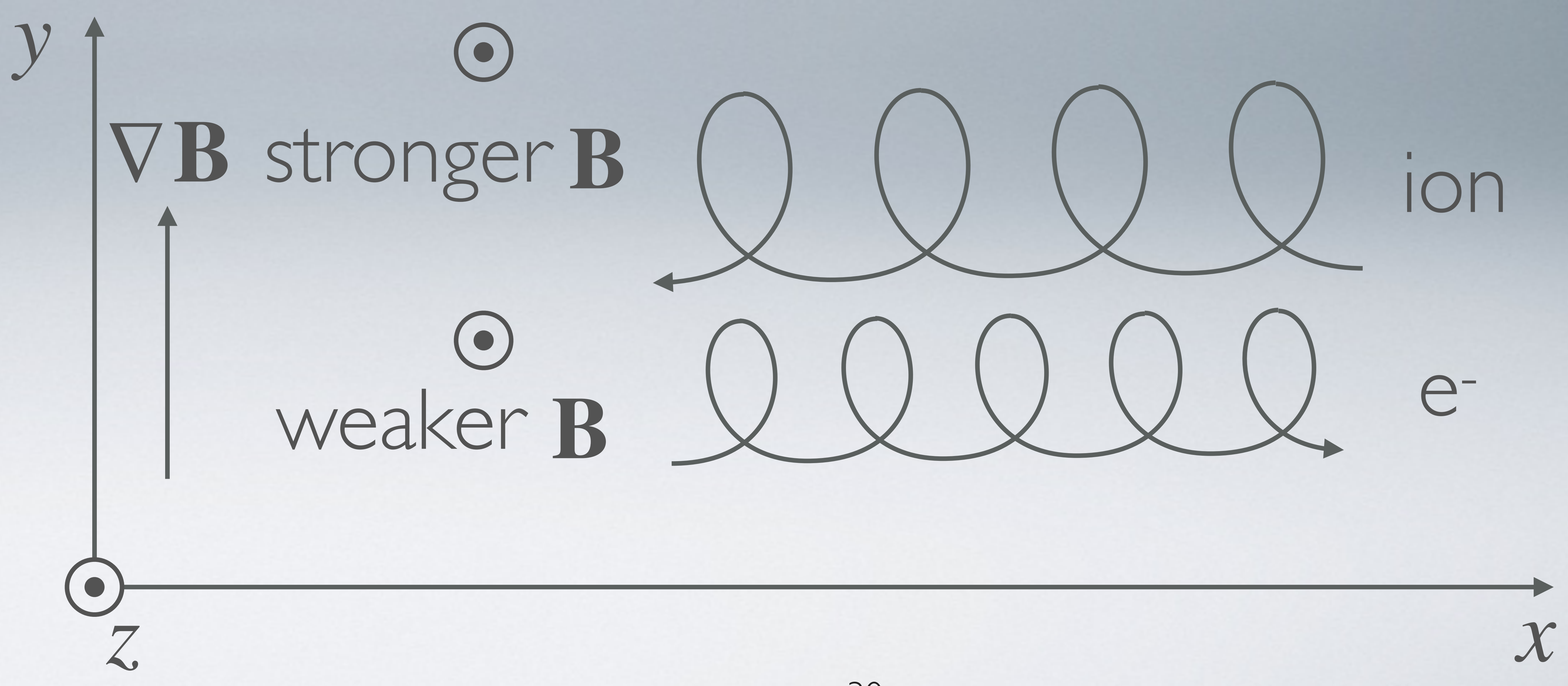
Field and gradient at the centre of the gyration



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Breakdown of Frozen-in approximation

Points to note about the grad-B drift:

Direction is dependent on charge so ions and electrons drift in opposite directions - drives a current

$$\mathbf{v}_{\nabla B} = \frac{W_\perp}{qB^3}\mathbf{B} \times \nabla \mathbf{B}$$

Field and gradient at the centre of the gyration

Proportional to $|\nabla \mathbf{B}|$, so zero for uniform field, becomes stronger as the spatial scale across B decreases

Proportional to particle energy $W_\perp$, so becomes more important as particle energy increases

Breakdown of Frozen-in approximation

A related drift occurs when the magnetic field is curved, i.e. its direction changes along $\mathbf{B}$. This is called the curvature drift.

Drift velocity:

$$\boxed{\mathbf{v}_{\mathrm{curv}} = \frac{2W_{\|}}{qB^3}\mathbf{B} \times \left[\left(\frac{\mathbf{B}}{B} \cdot \nabla \right) \mathbf{B} \right]}$$

For solar system plasmas, at thermal (eV-keV) energies, $(E/B) \gg |\mathbf{v}_{\nabla \mathbf{B}}|, |\mathbf{v}_{\mathrm{curv}}|$ so the frozen in approximation is reasonably valid for solar system

At high (MeV) energies particle drifts dominate and F-i-F breaks down

Where the spatial scale of gradients becomes small (e.g. near boundaries such as the magnetopause) F-i-F even at low energies

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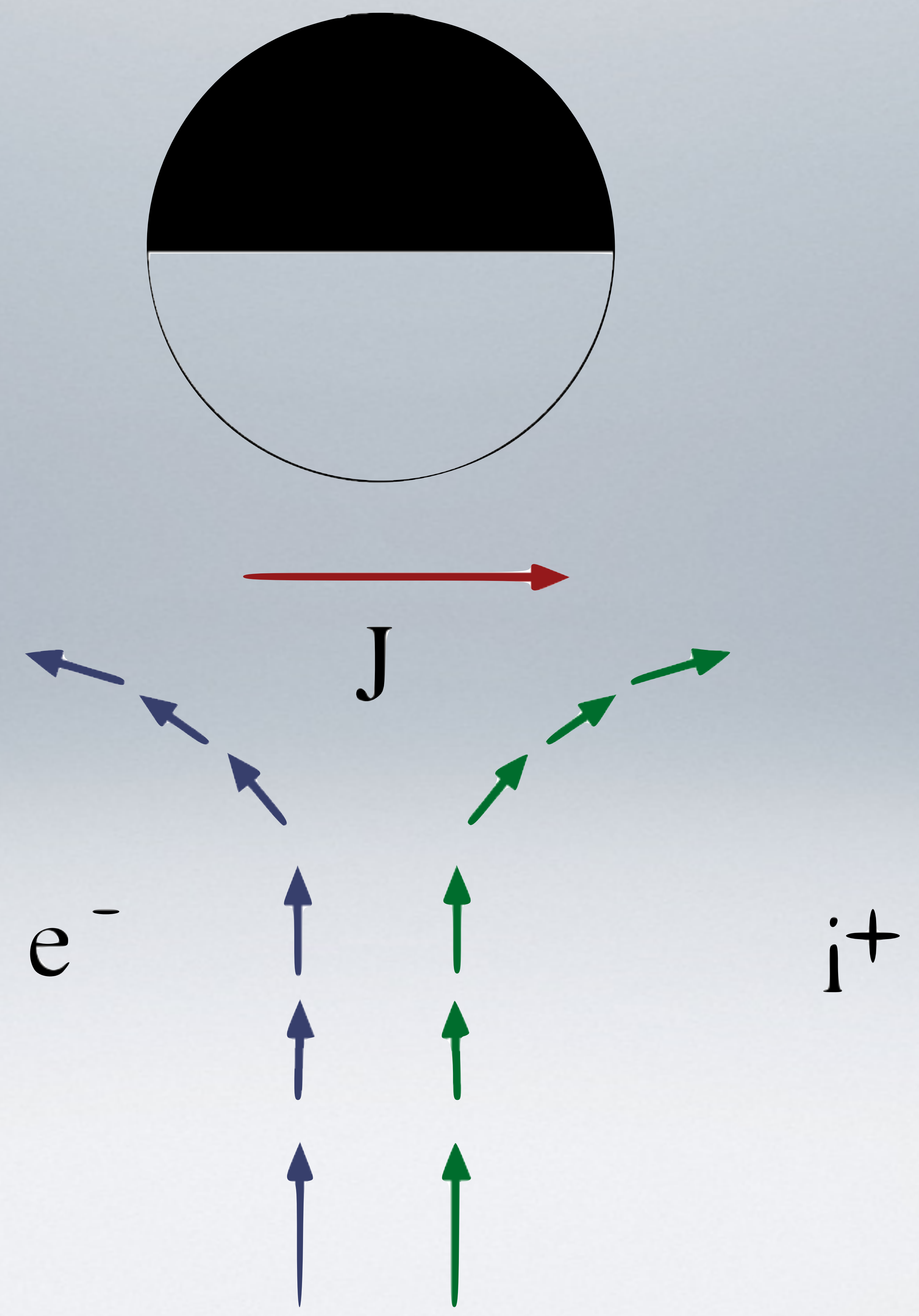
Breakdown of Frozen-in approximation



Effect of collisions

We have seen that the effect of a perpendicular electric field $E_\perp$ in a collisionless plasma is to cause both ions and electrons to drift with the same velocity $\mathbf{v}_E = \mathbf{E}\times\mathbf{B}/B^2$ perpendicular to both E and B.

Electric currents are known to transfer stress between a planet and its magnetosphere, yet principles of electromagnetism state that electric currents only do work when the condition $\mathbf{j}\cdot\mathbf{E} > 0$ is met.

The current is parallel to the electric field

How can this be? In the ionosphere, collisions with neutrals enable conductivity in the direction of the electric field

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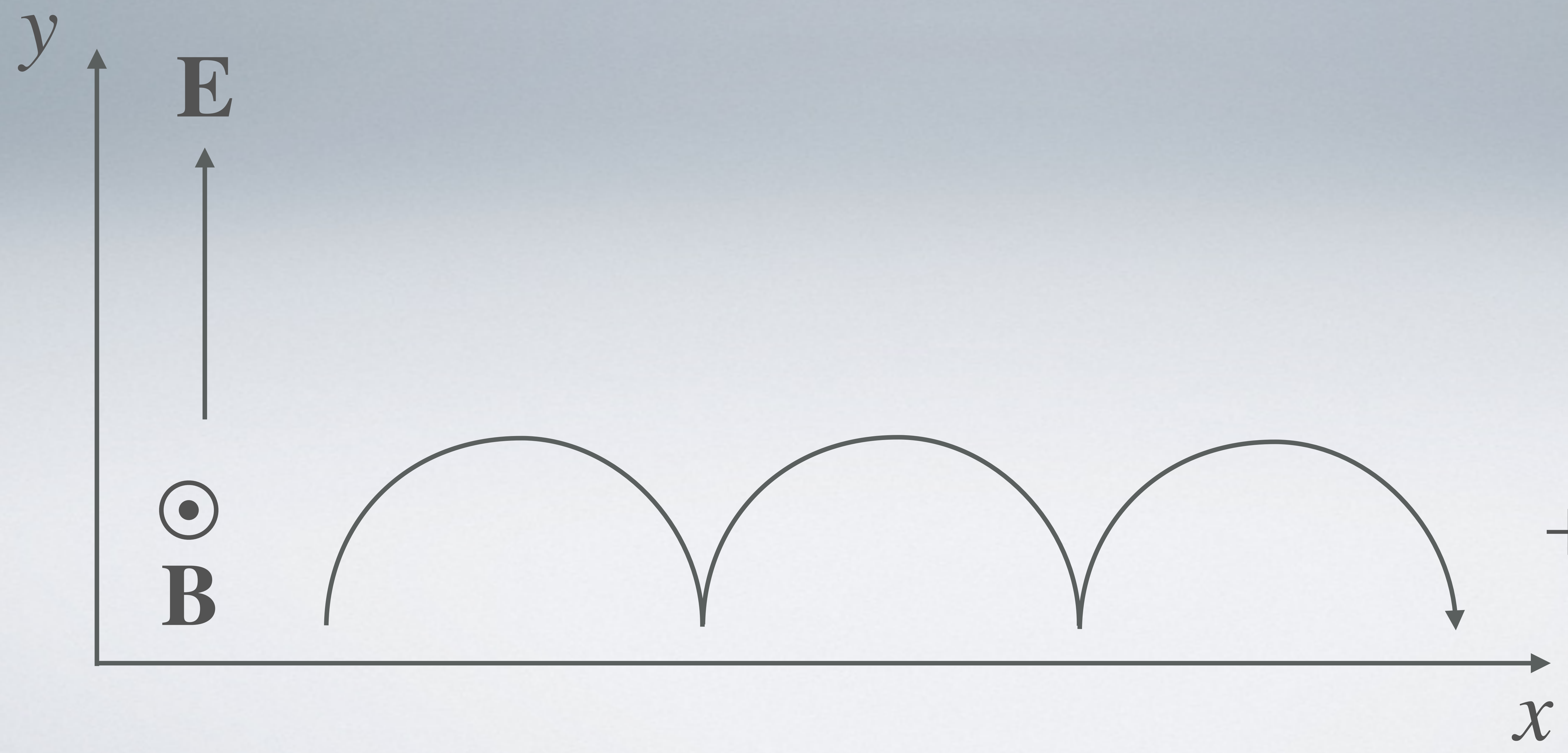
Effect of collisions

If we take $\mathbf{B} = B\,\underline{\hat{\mathbf{z}}}$ as before and $\mathbf{E} = E_\perp\,\underline{\hat{\mathbf{y}}}$:

Imposed e.g. by magnetospheric dynamics

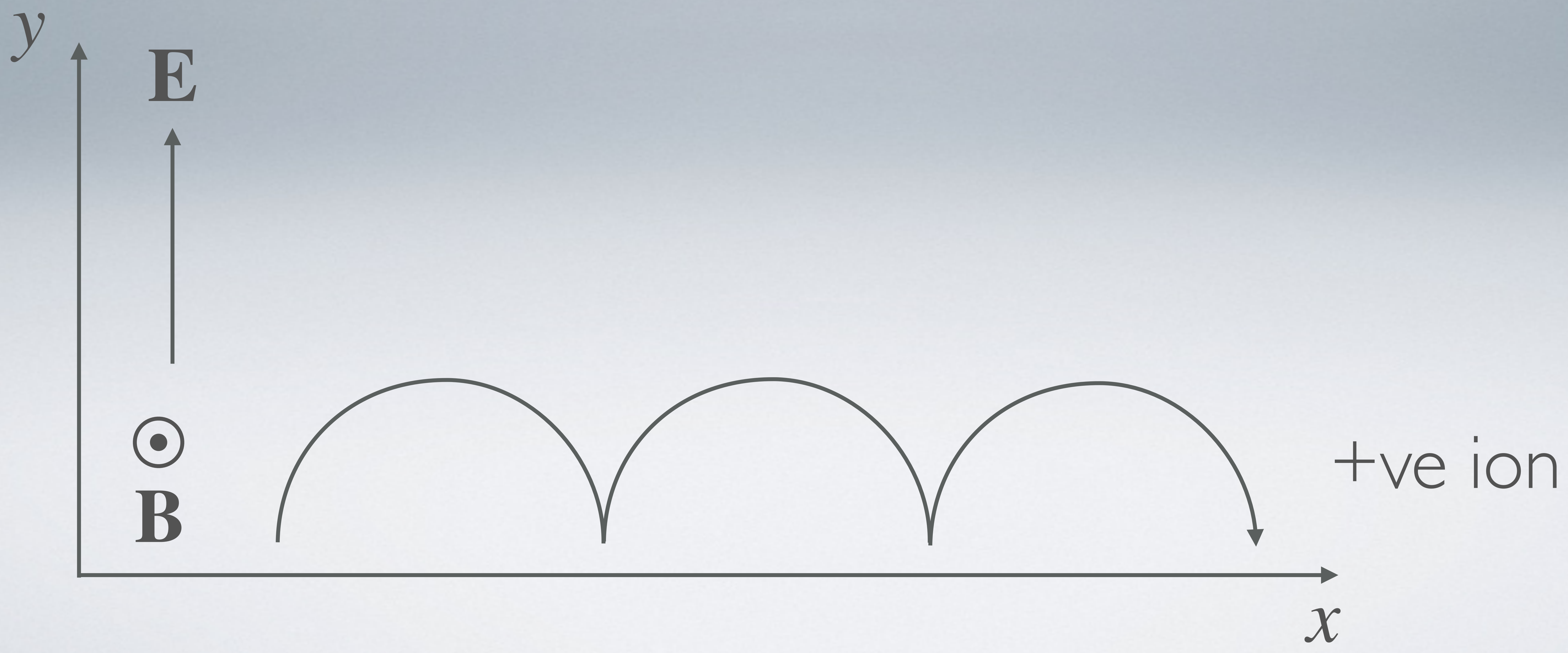
Considering a collisionless plasma, particles will simply gyrate and $\mathbf{E} \times \mathbf{B}$ drift in the $+\hat{\mathbf{x}}$ direction



Effect of collisions

Introducing collisions with neutrals, at all times there is a chance that the particle will experience a collision and receive a random Δv

In the absence of a bulk neutral wind, on average these Δv s sum to zero, i.e. to a first approximation, particles are brought to a halt by each collision.

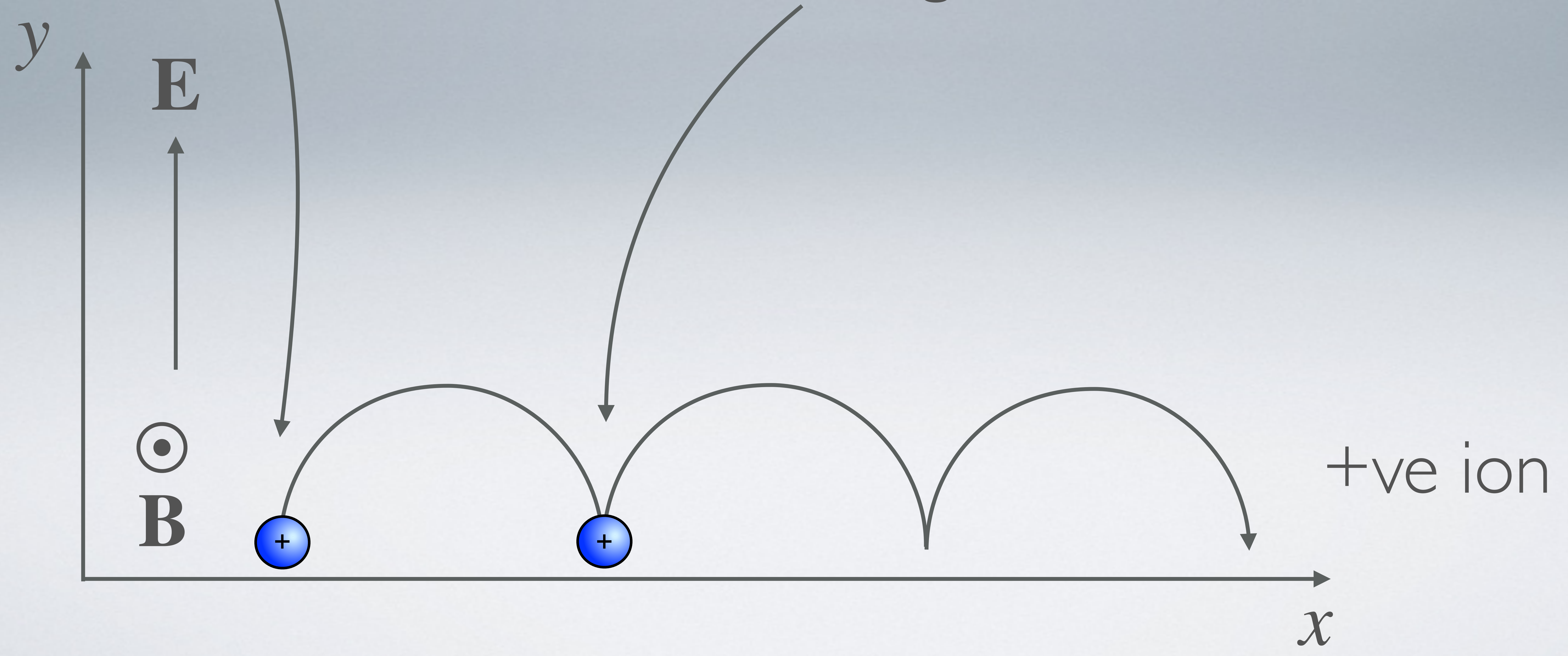


Effect of collisions

If the collision frequency exactly equals the gyrofrequency, collisions have no effect

A particle at rest here…

…completes one gyration and comes to rest again at the time of the next collision



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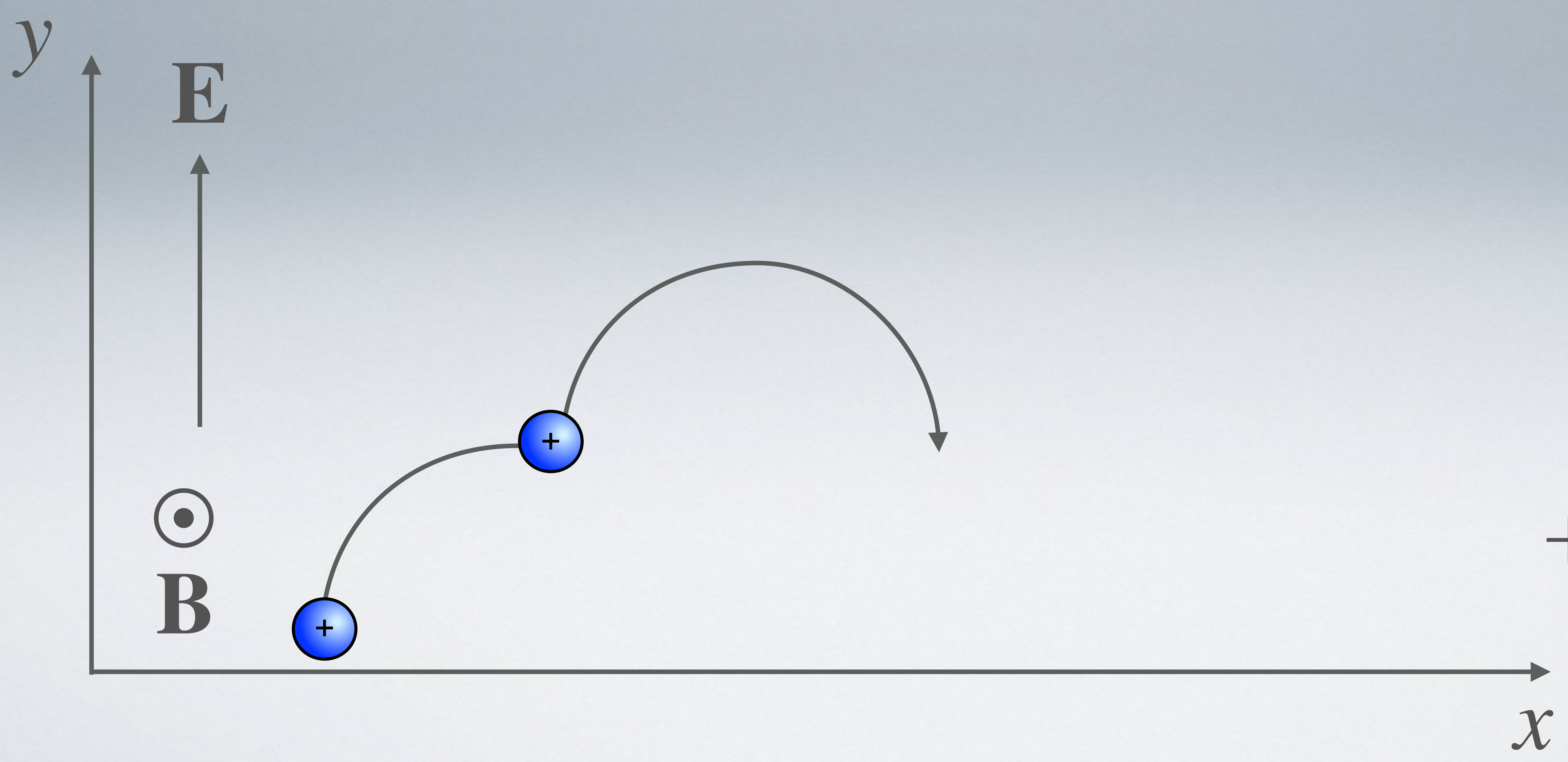
Effect of collisions

If the collision frequency is greater than the gyrofrequency, collisions bring the particle to a halt at a random time during its gyration, whereupon it will recommence $\mathbf{E} \times \mathbf{B}$ drifting owing to the electric field, but from a starting point further in the $+\hat{\mathbf{y}}$ direction



+ve ion

Effect of collisions

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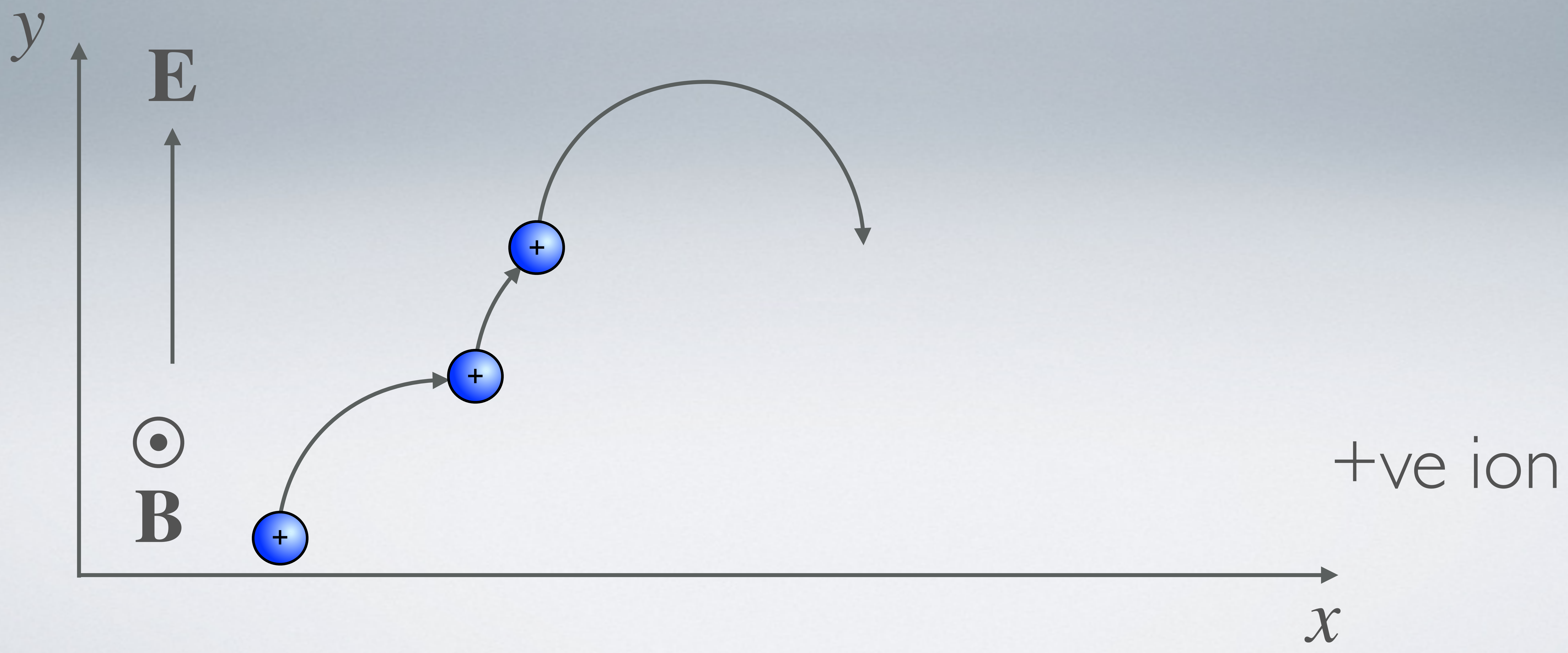

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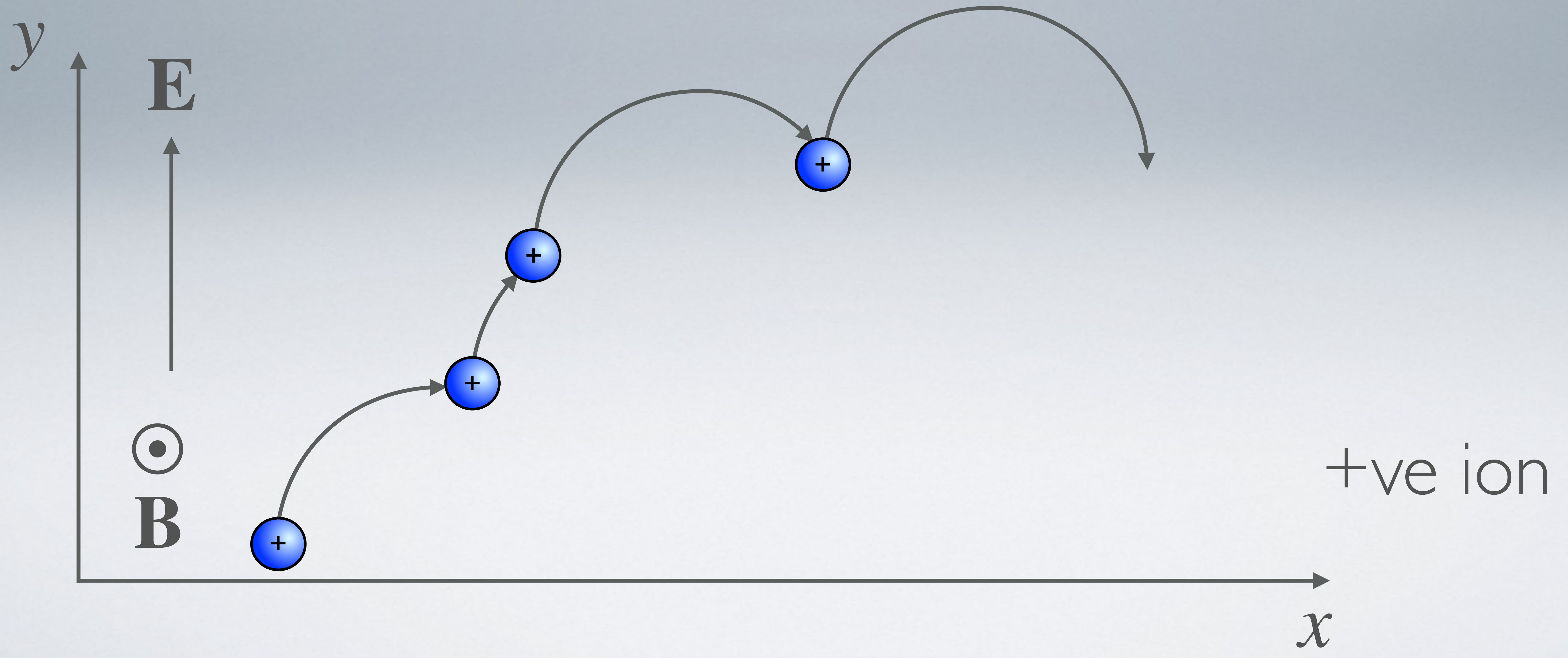
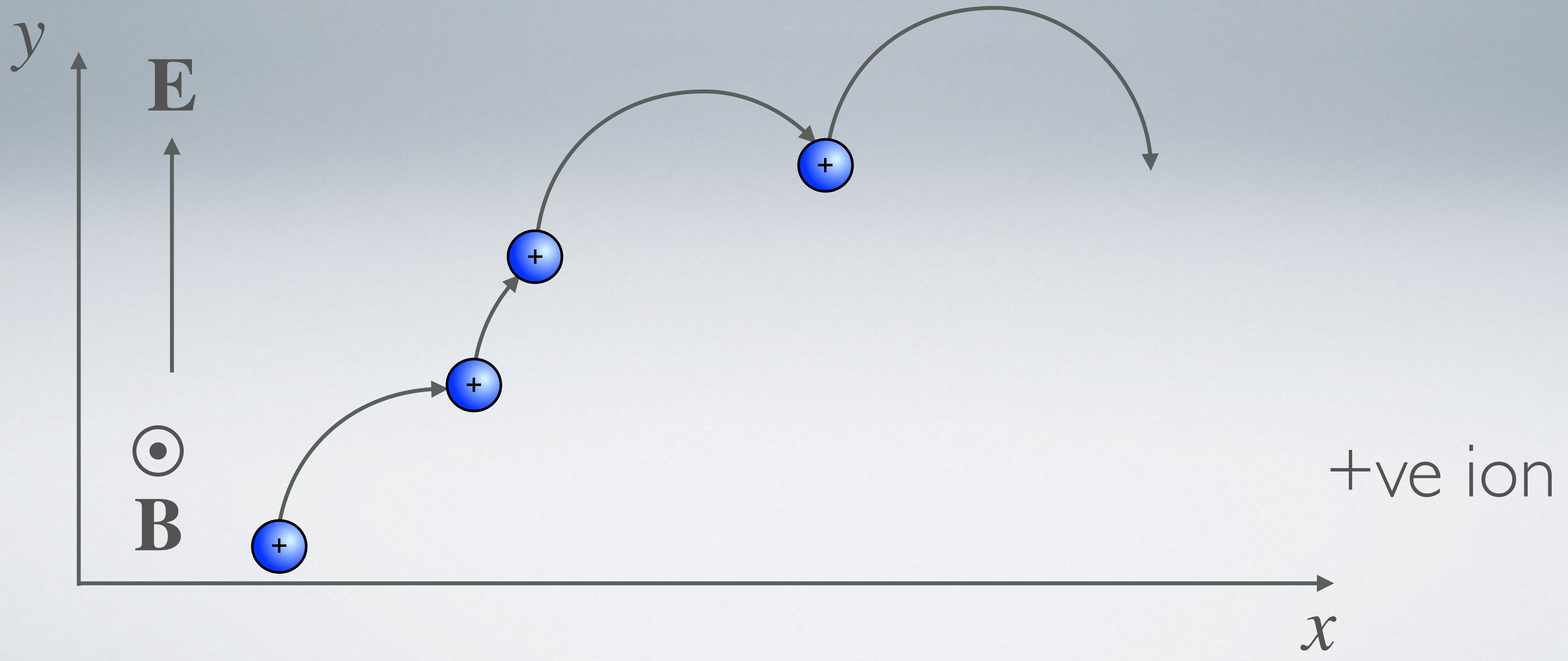


Effect of collisions

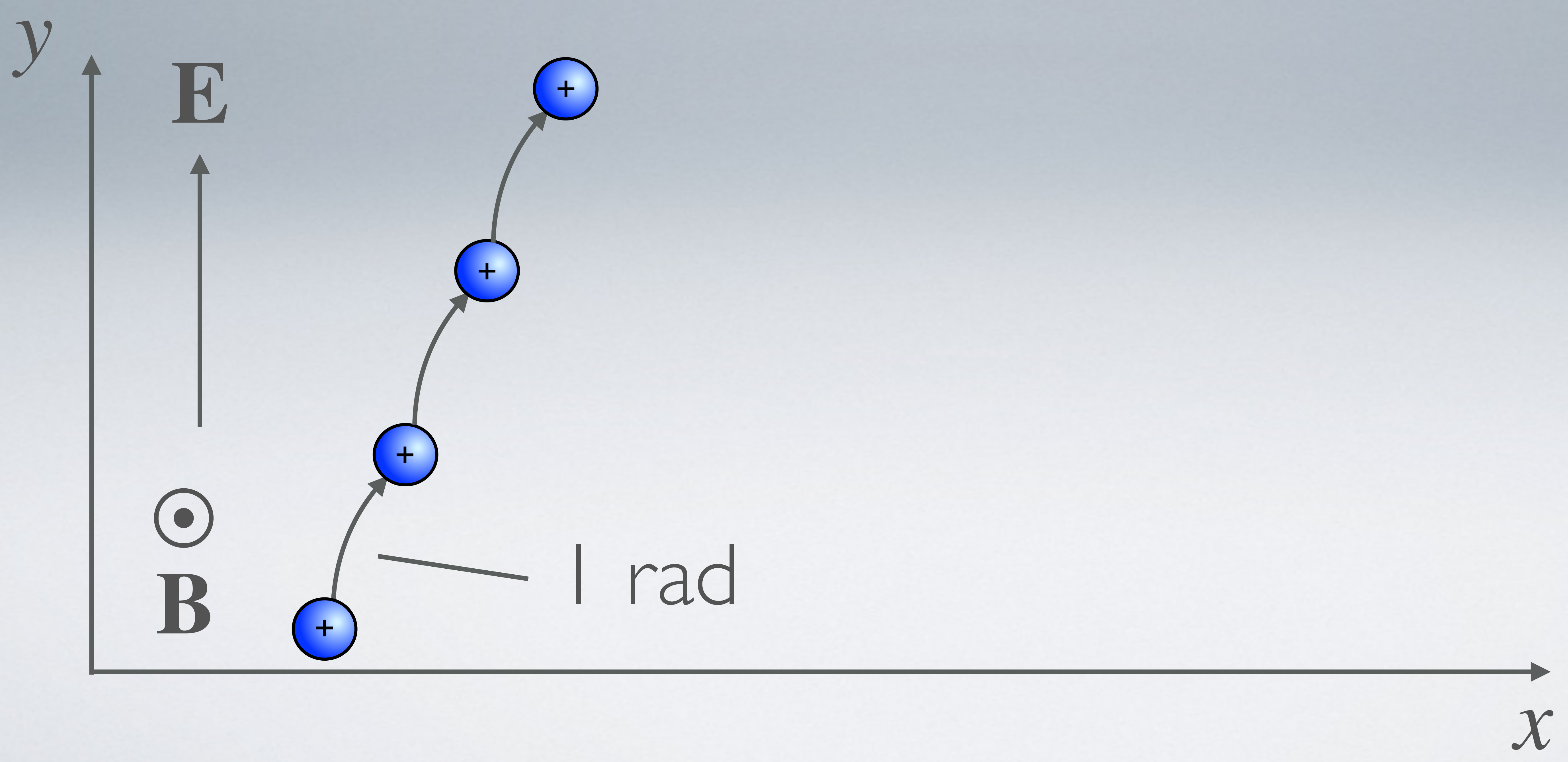
This mobility in the direction of the electric field enables Pedersen conductivity and the resulting current in this direction is a Pedersen current

Mobility in the $+\hat{\mathbf{x}}$ direction is reduced



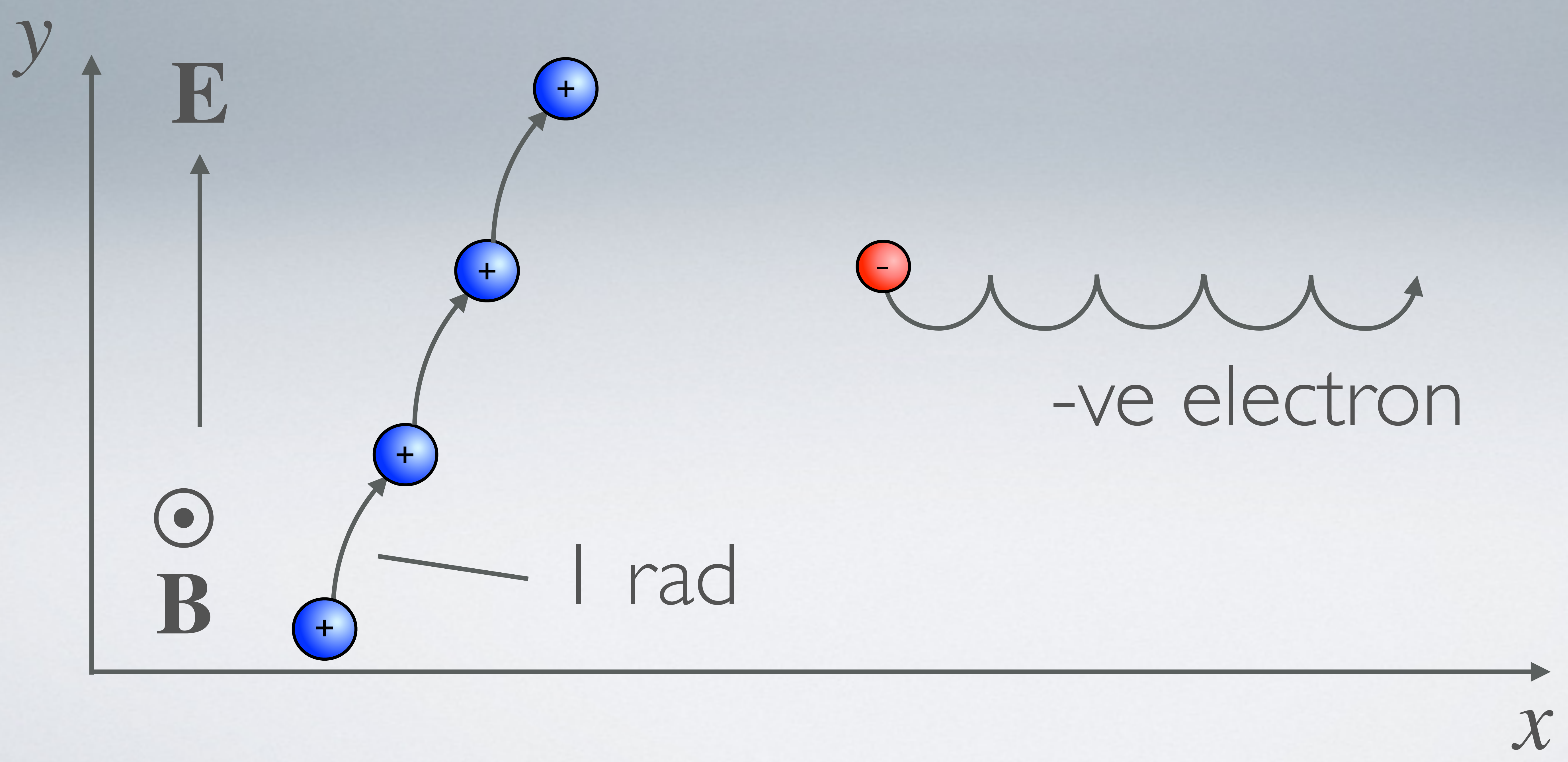
Effect of collisions

Pedersen conductivity is maximised when the collision frequency ν (in collisions per second) is equal to the gyrofrequency Ω in radians per second, such that a particle travels 1 radian of arc between each collision



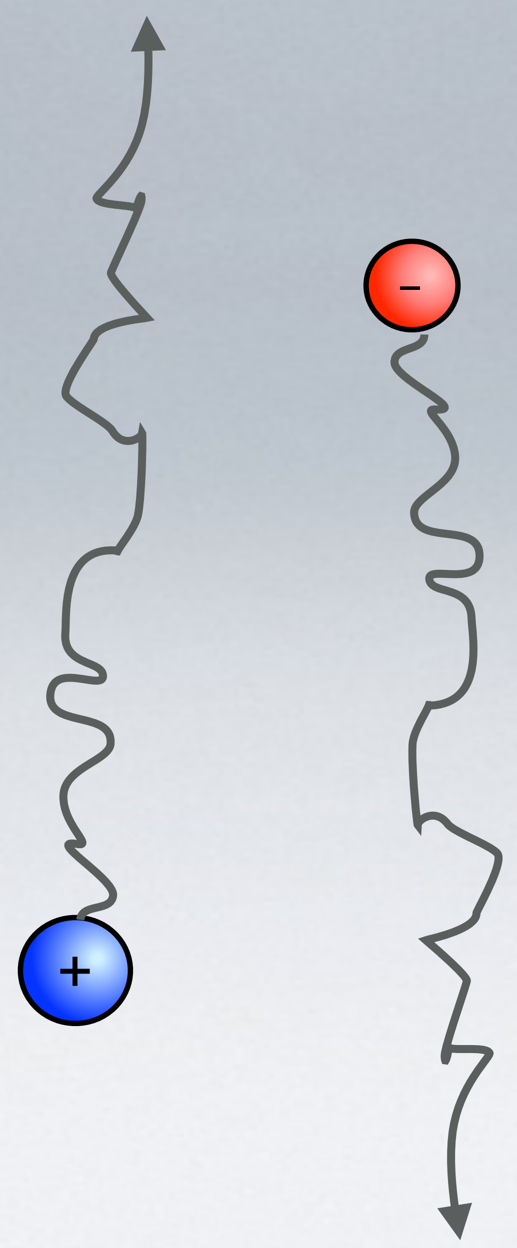
Effect of collisions

Ions have larger collisional cross sections than electrons, leading to higher collision frequencies in general. This results in reduced ion mobility in the $\hat{\mathbf{x}}$ direction, and a corresponding differential flow of charge - a current. This is termed the Hall current as it flows perpendicular to the electric field (and hence does no work)



Effect of collisions

If the collision frequency is much higher than the gyrofrequency, particles tend to slowly drift in the direction of the electric field. Eventually, particles lose all mobility once collisions dominate particle motion



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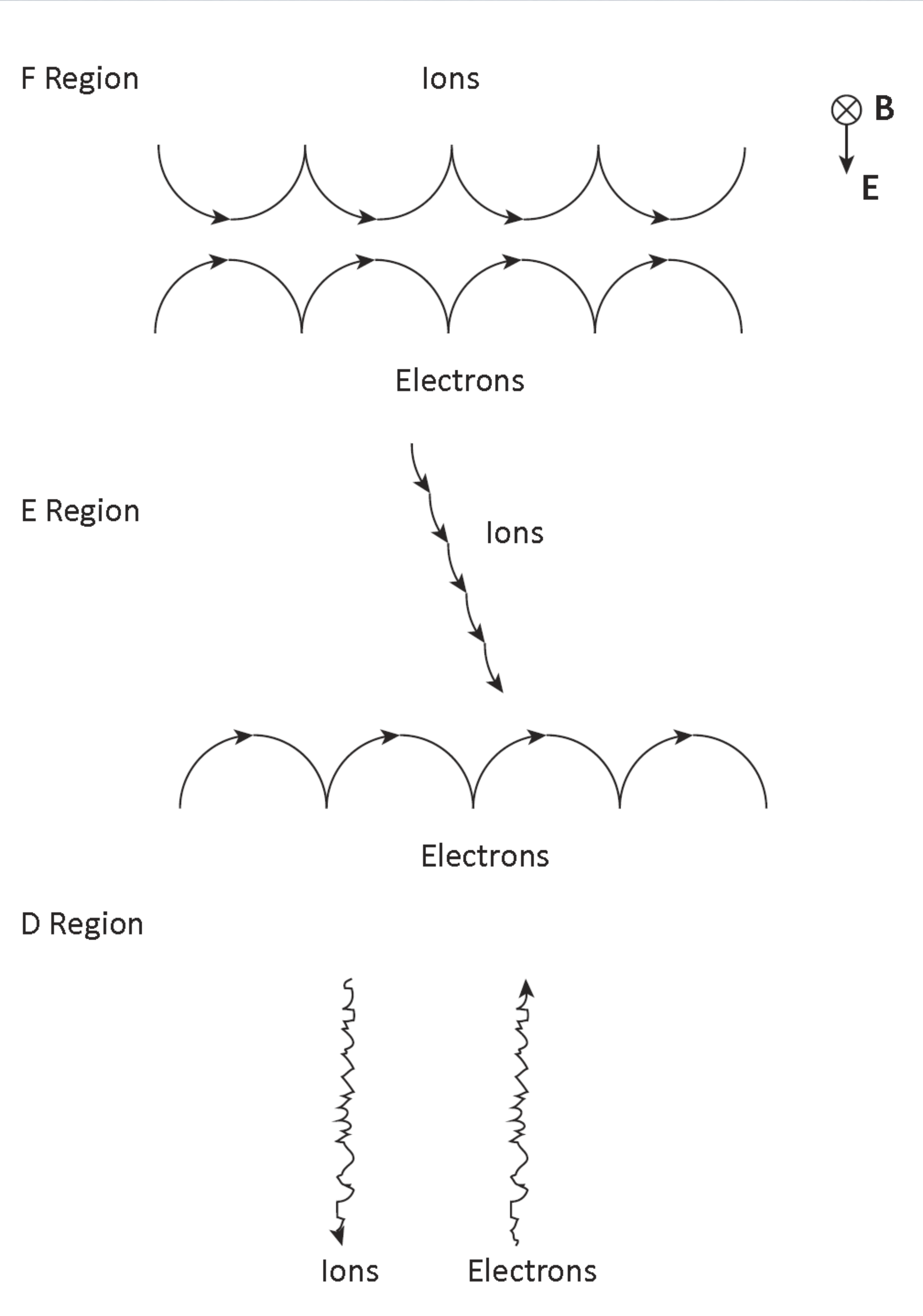
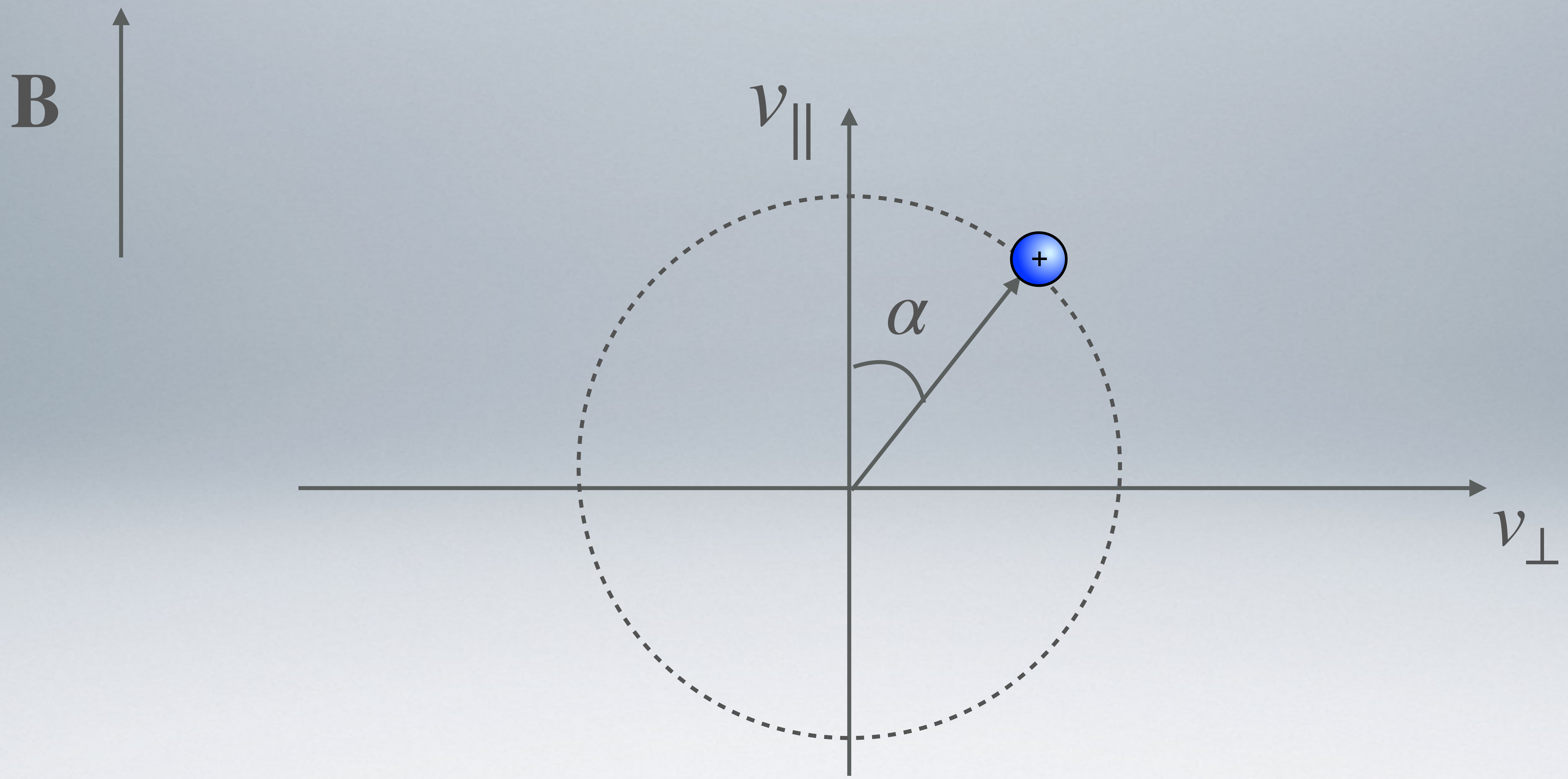
Effect of collisions

Altitude



Velocity space



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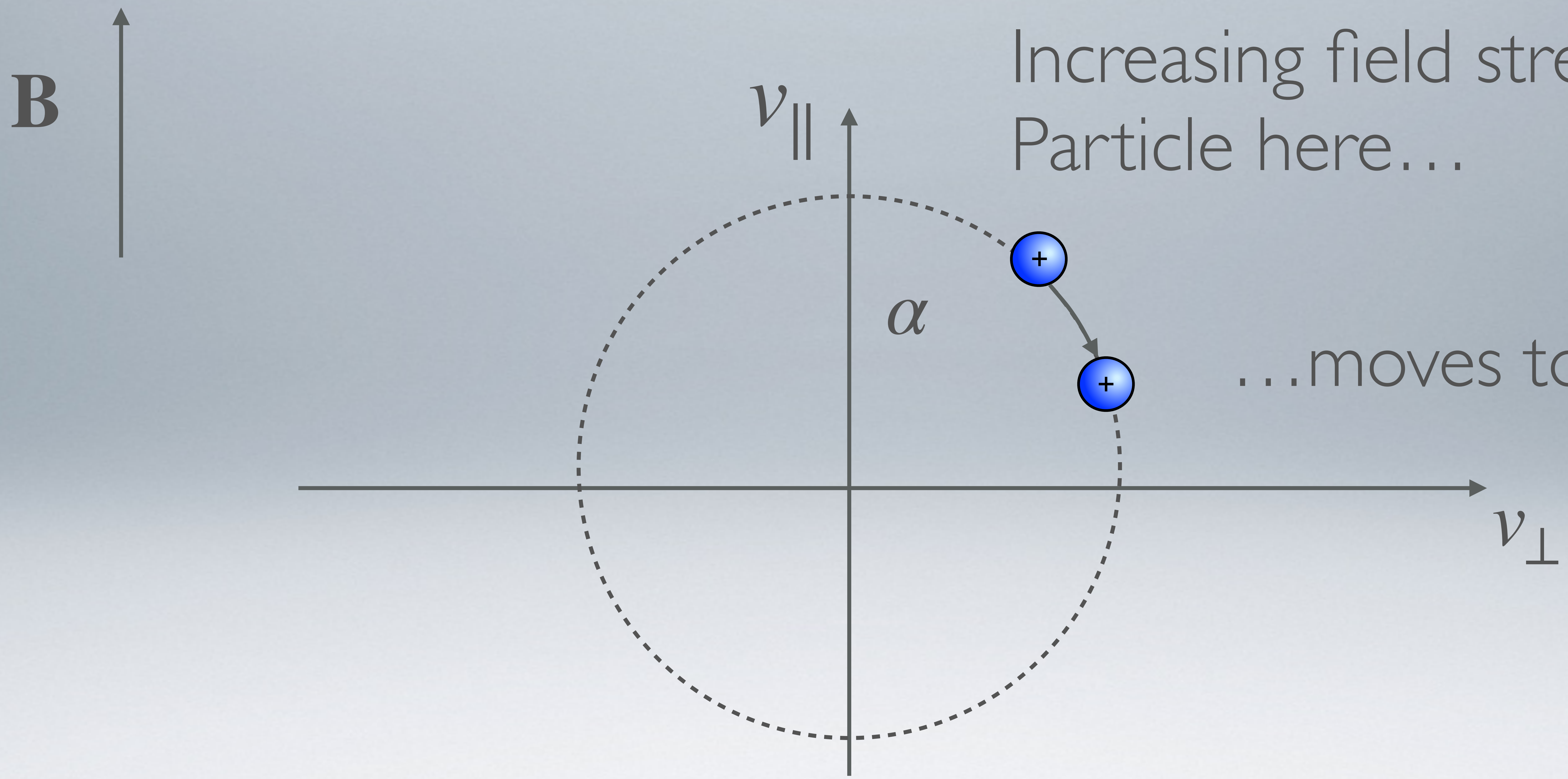
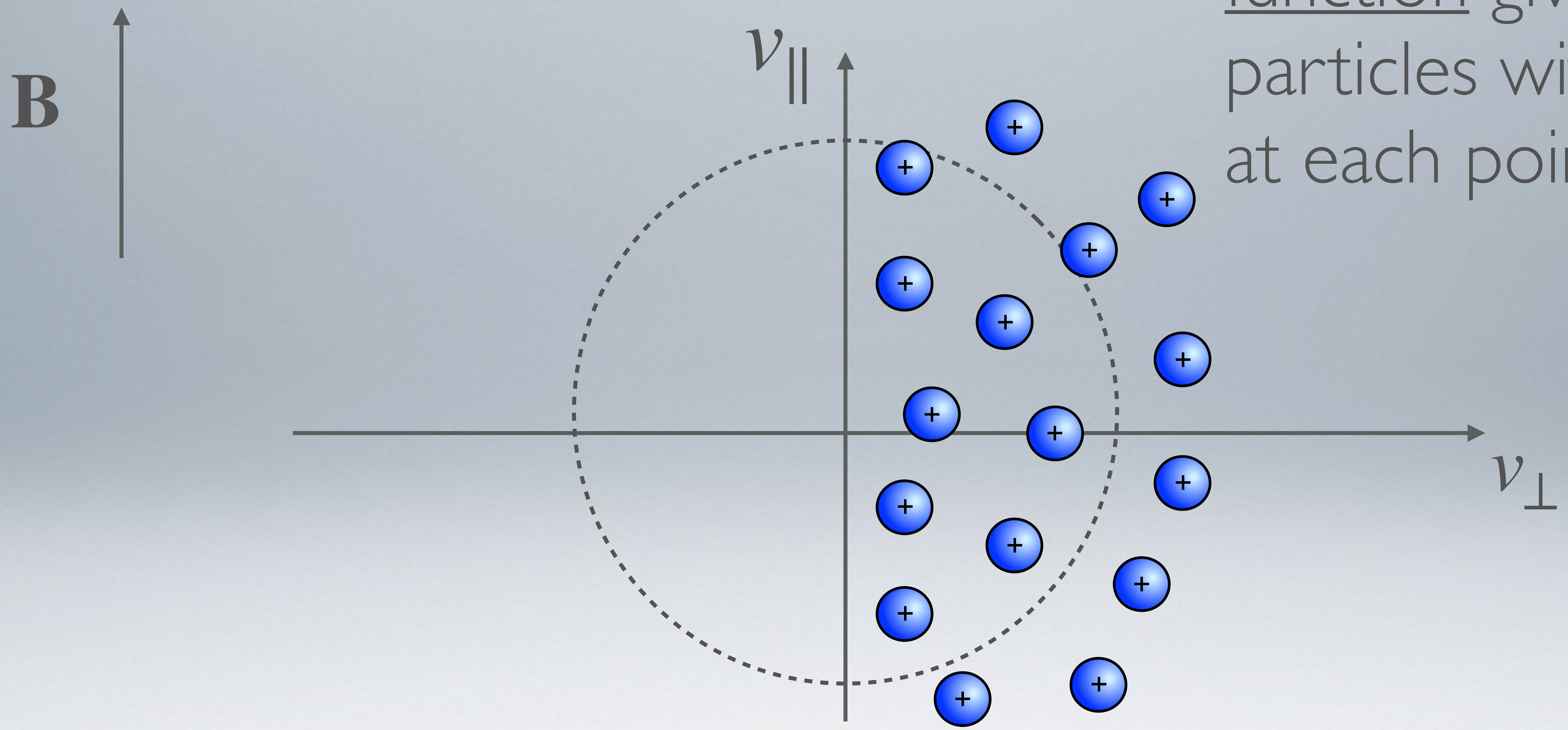
Velocity space



Increasing field strength
Particle here…

…moves to here

Velocity space



The particle distribution function gives the number of particles with a given velocity at each point in (real) space

This distribution is isotropic

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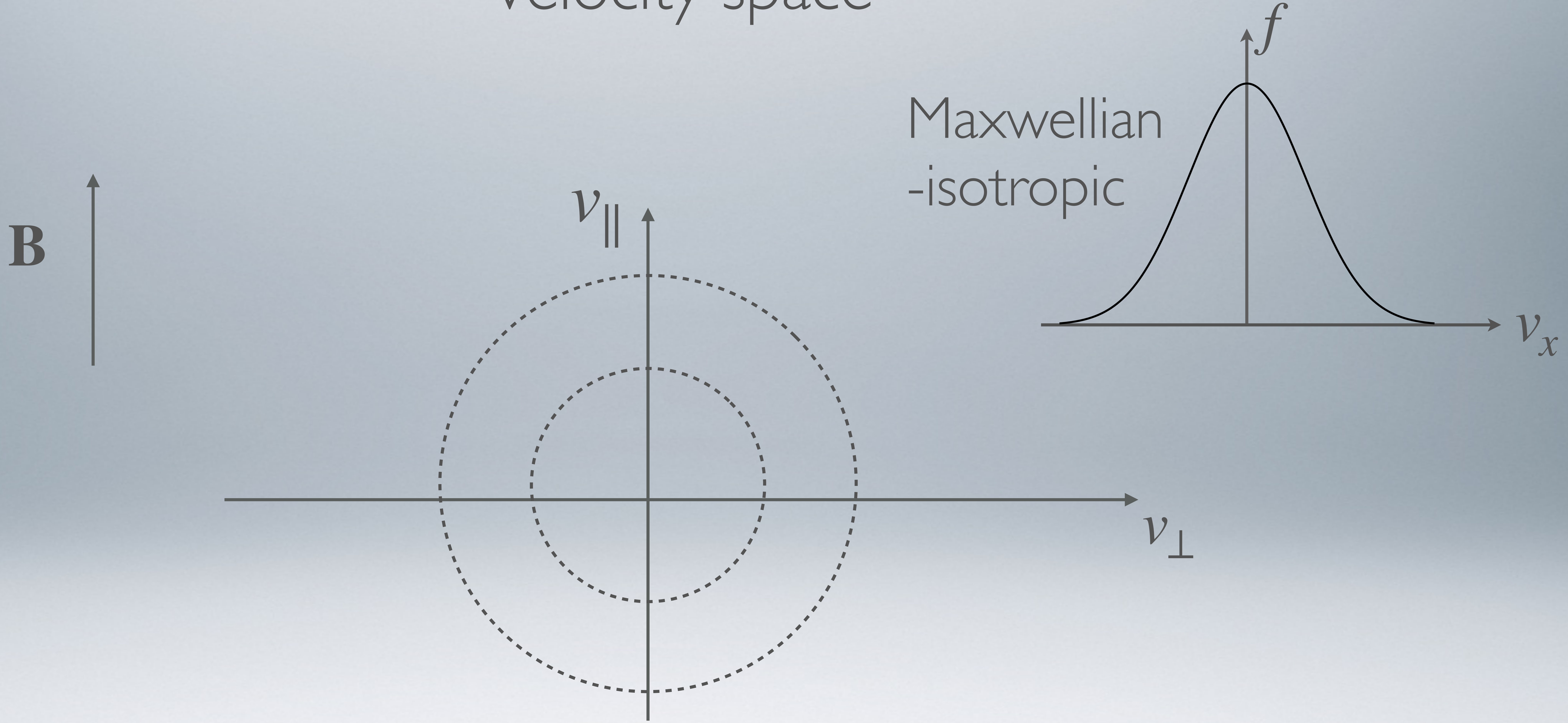
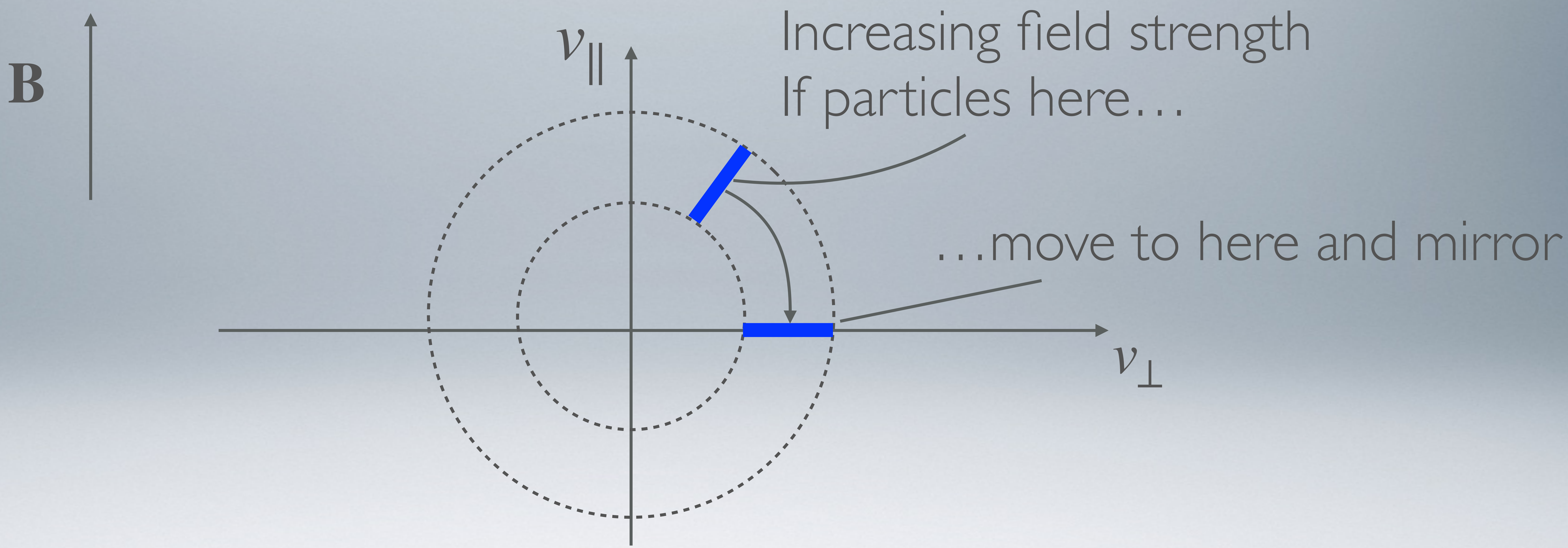
Velocity space

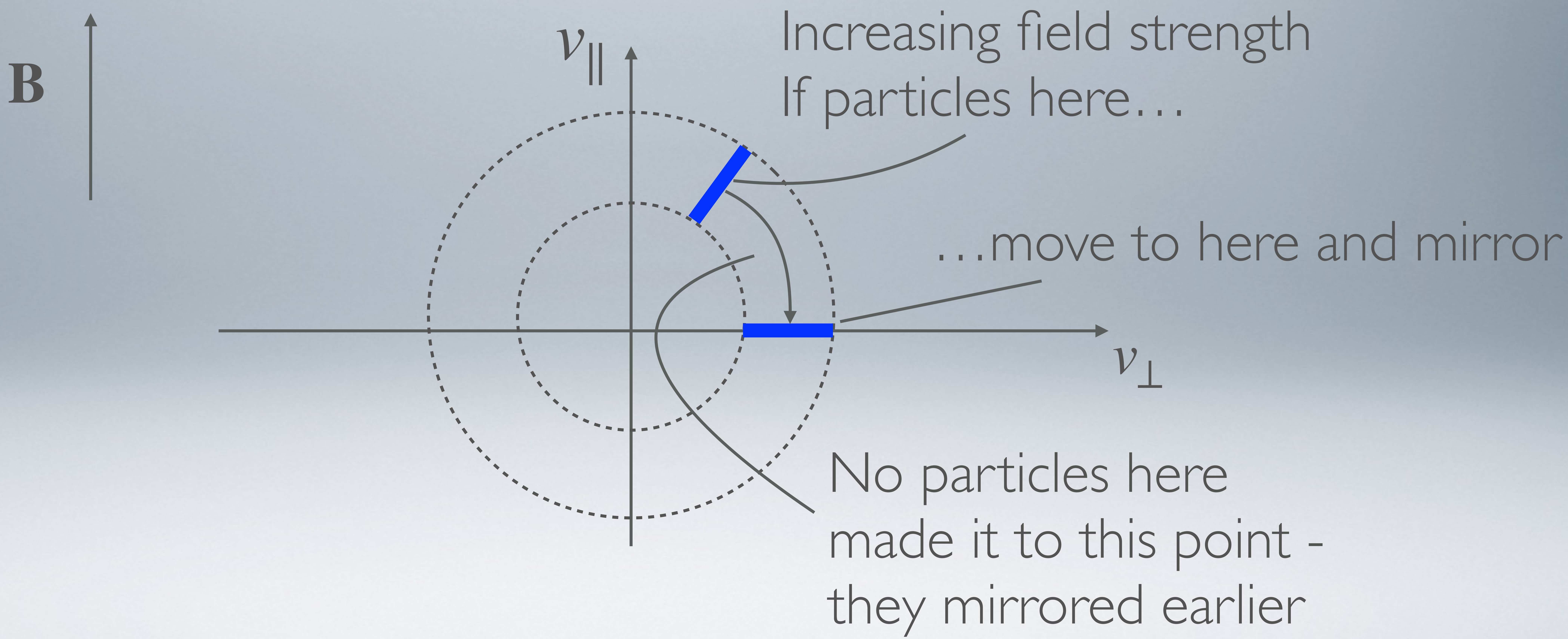


Although space plasmas are collisionless, wave-particle interactions tend to produce Maxwellian-like distributions

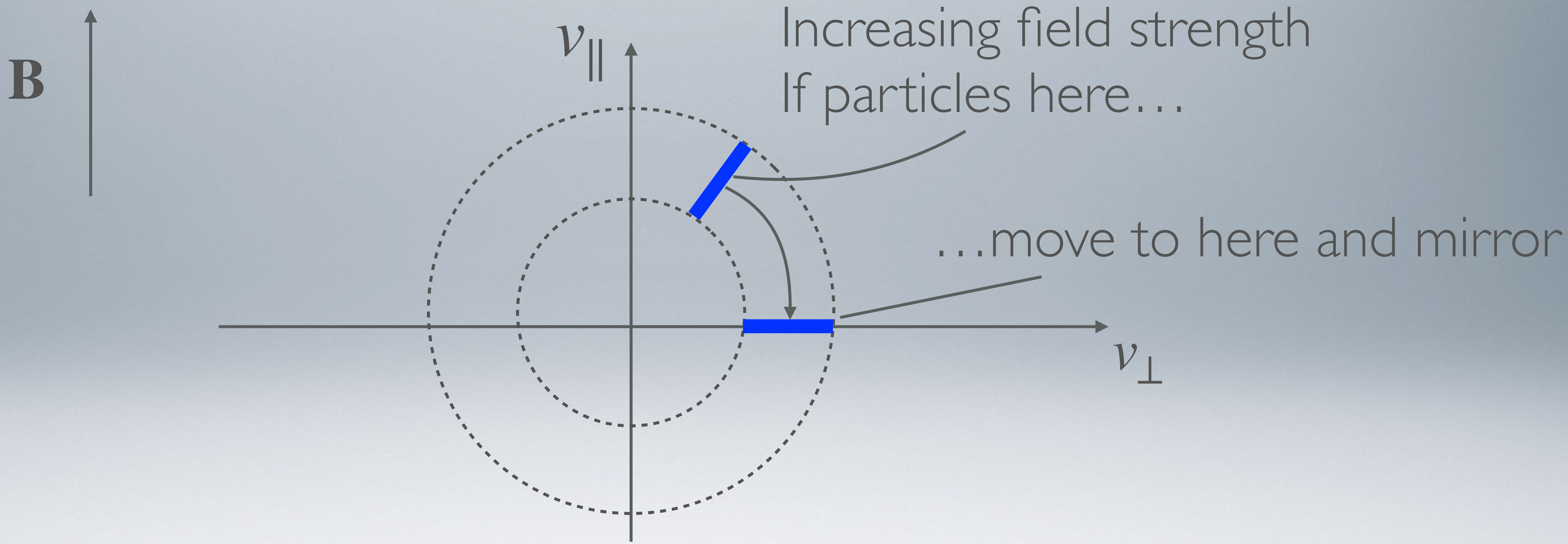
Velocity space



Velocity space



Velocity space



Isotropic particle distribution functions are invariant with field strength
-implication: number density and pressure constant on a field line

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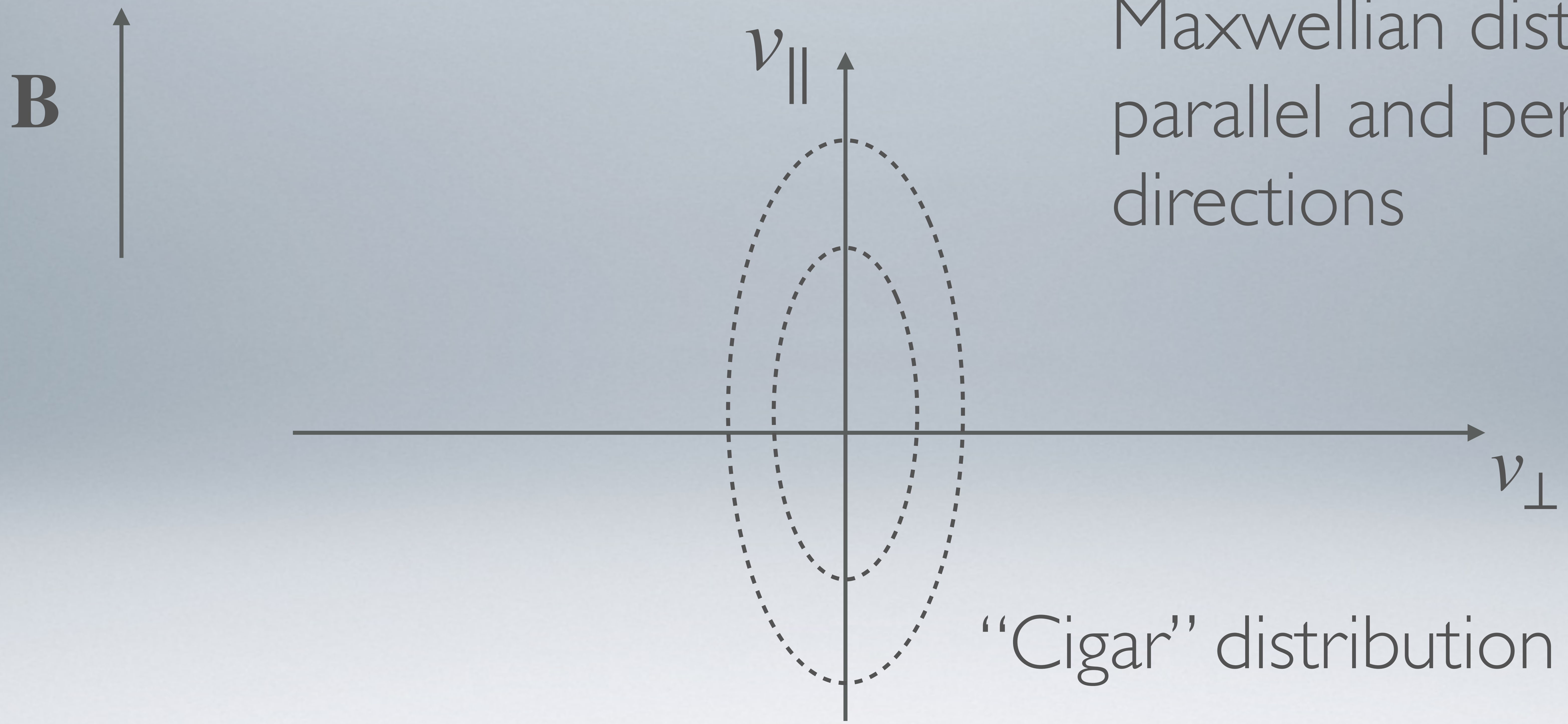
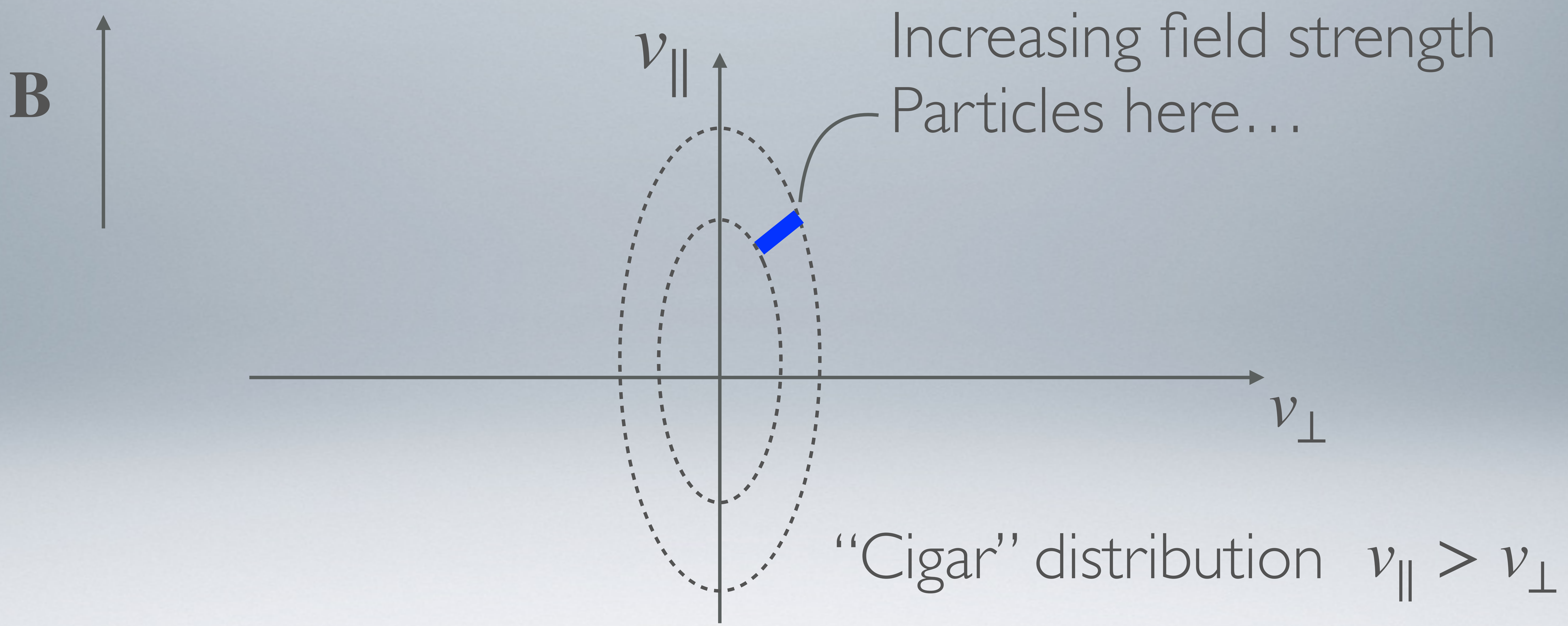
Velocity space



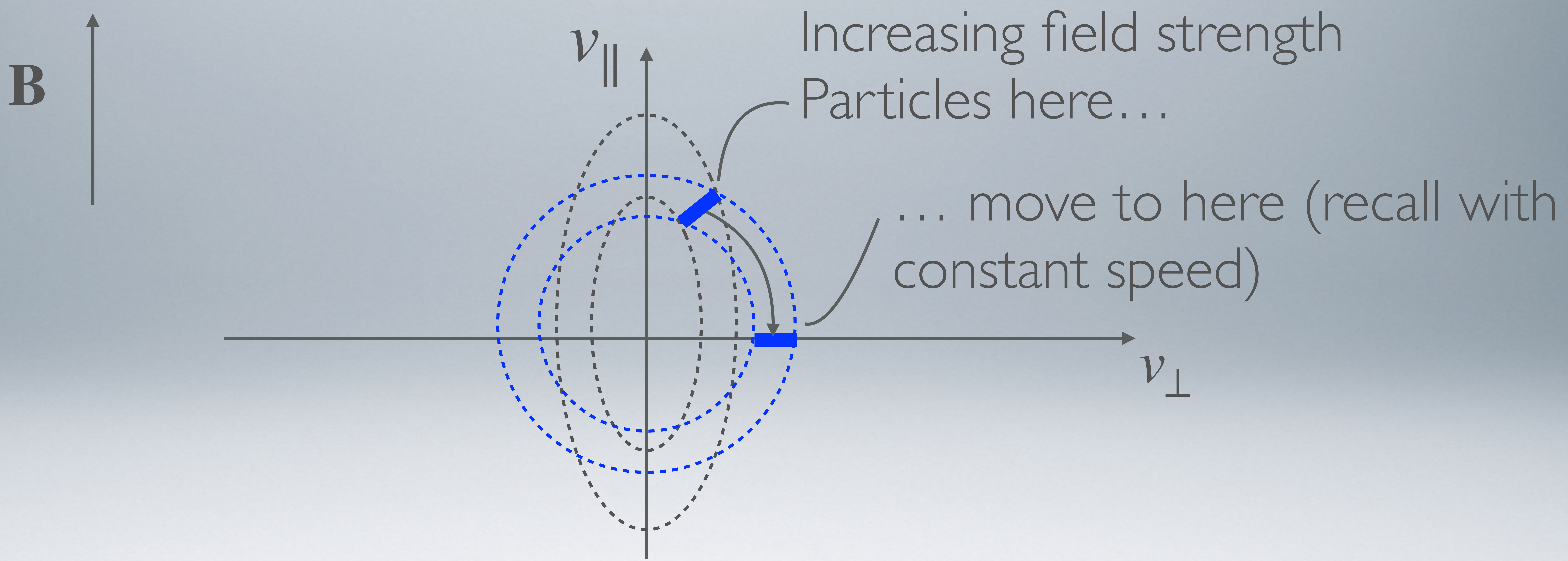
bi-Maxwellian - two separate Maxwellian distributions in parallel and perpendicular directions

"Cigar" distribution $v_{||} > v_\perp$

Velocity space

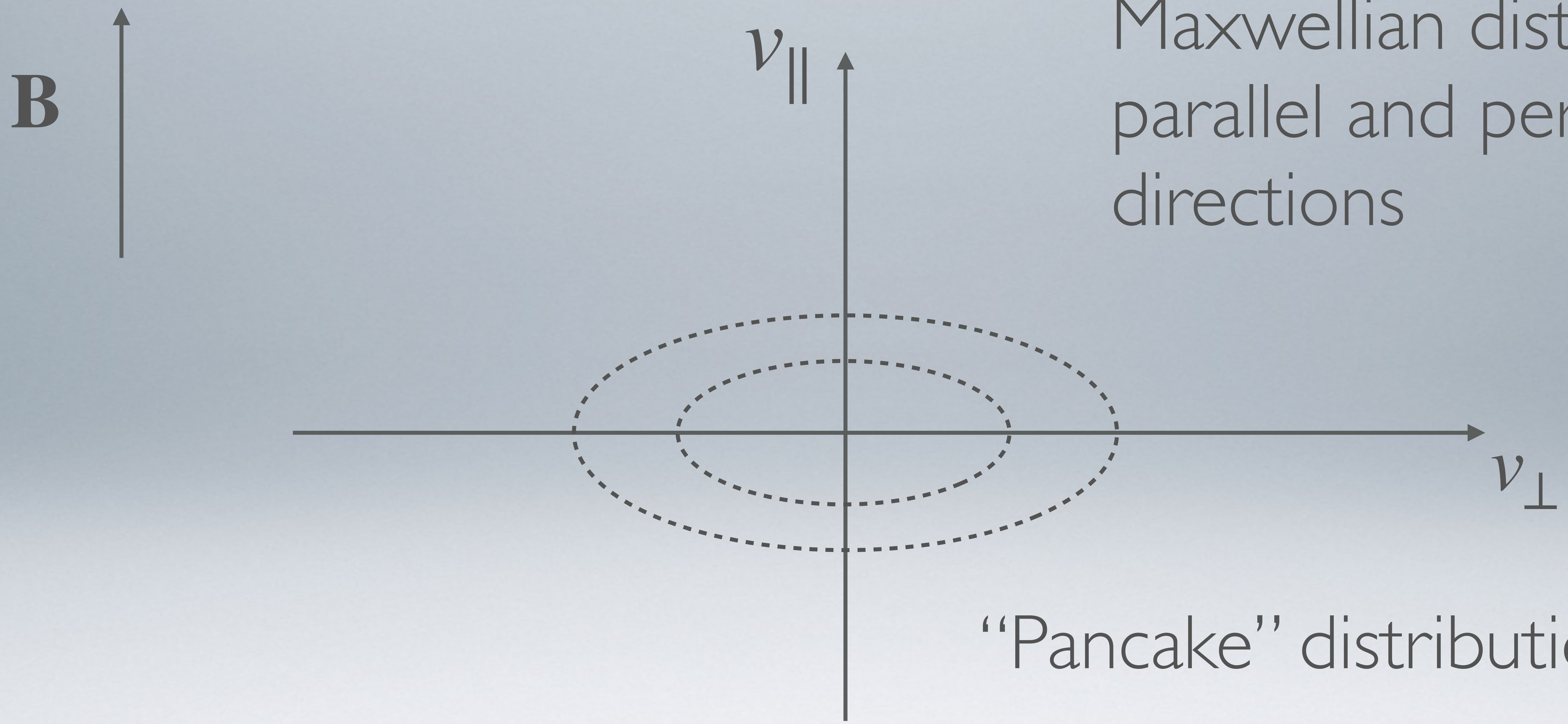


Velocity space



Increasing field strength *isotropises* the distribution to that shown in blue
Implication: number density and pressure increase with B

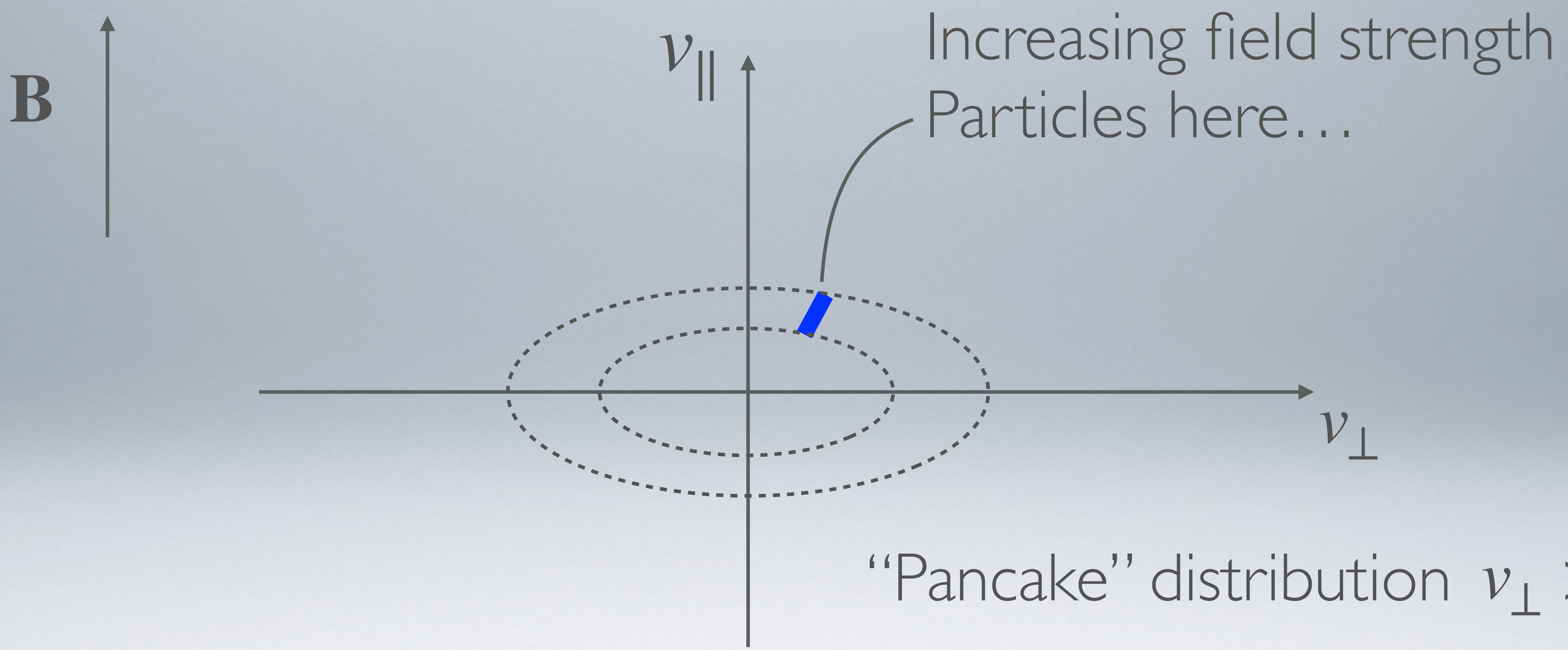
Velocity space



bi-Maxwellian - two separate Maxwellian distributions in parallel and perpendicular directions

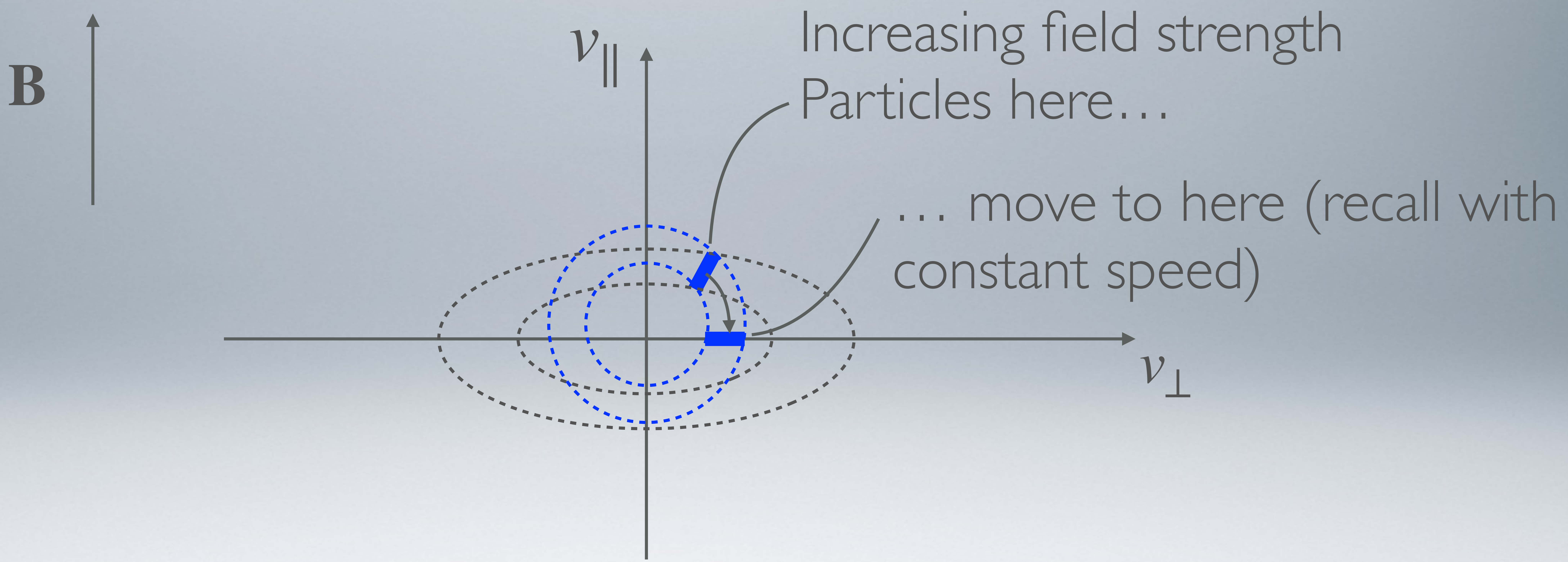
"Pancake" distribution $v_{\perp} > v_{\|}$

Velocity space



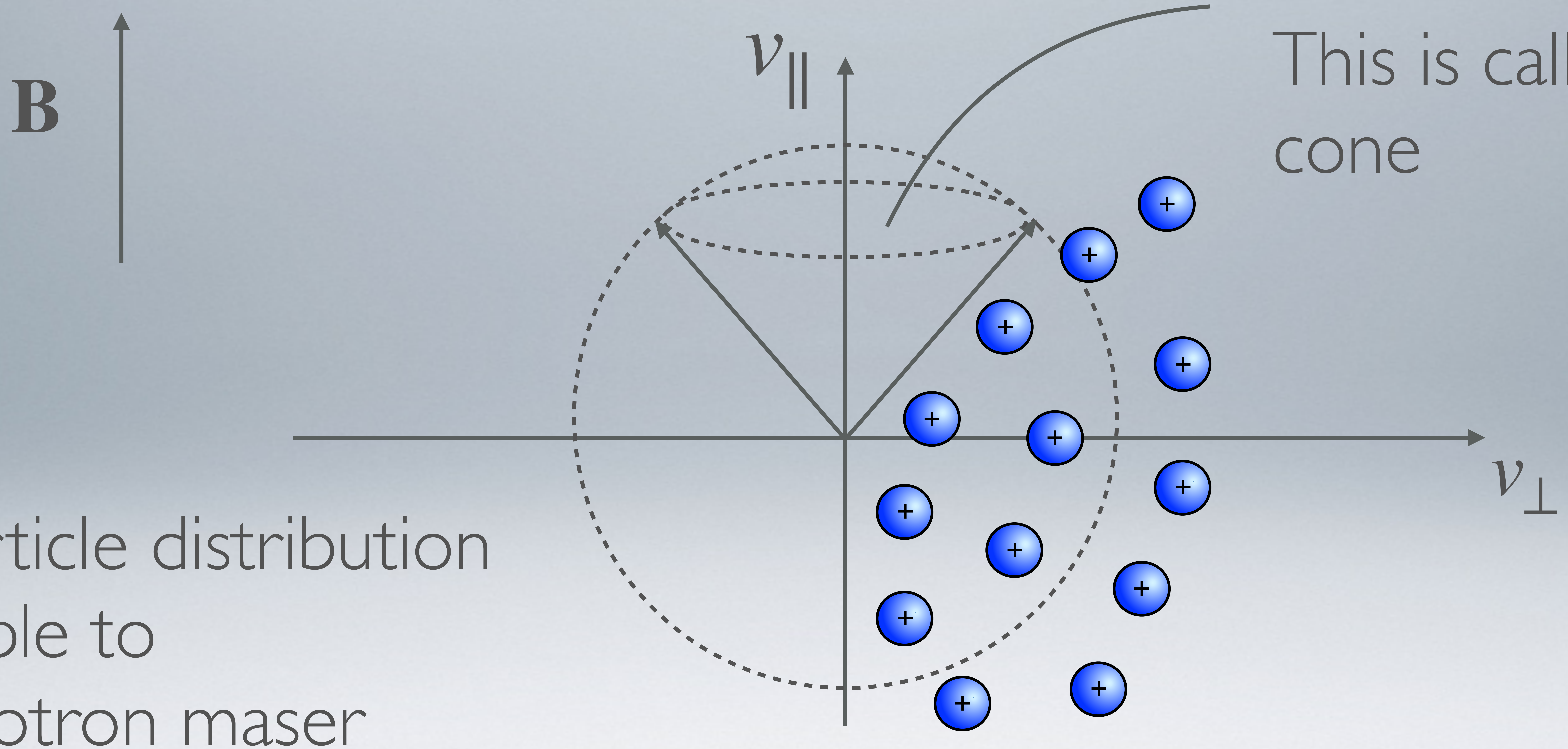
"Pancake" distribution $v_{\perp} > v_{\parallel}$

Velocity space



Again, increasing field strength *isotropises* the distribution to that shown in blue
Implication: number density and pressure decrease with B

Velocity space



This particle distribution is unstable to the cyclotron maser instability - drives radio emissions

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Task 1 - Simple Single Particle Motion solution

Change to the directory containing spm.py. Open the file spm.py

At the terminal type:

```
python spm.py
```

Is the result as we expect?

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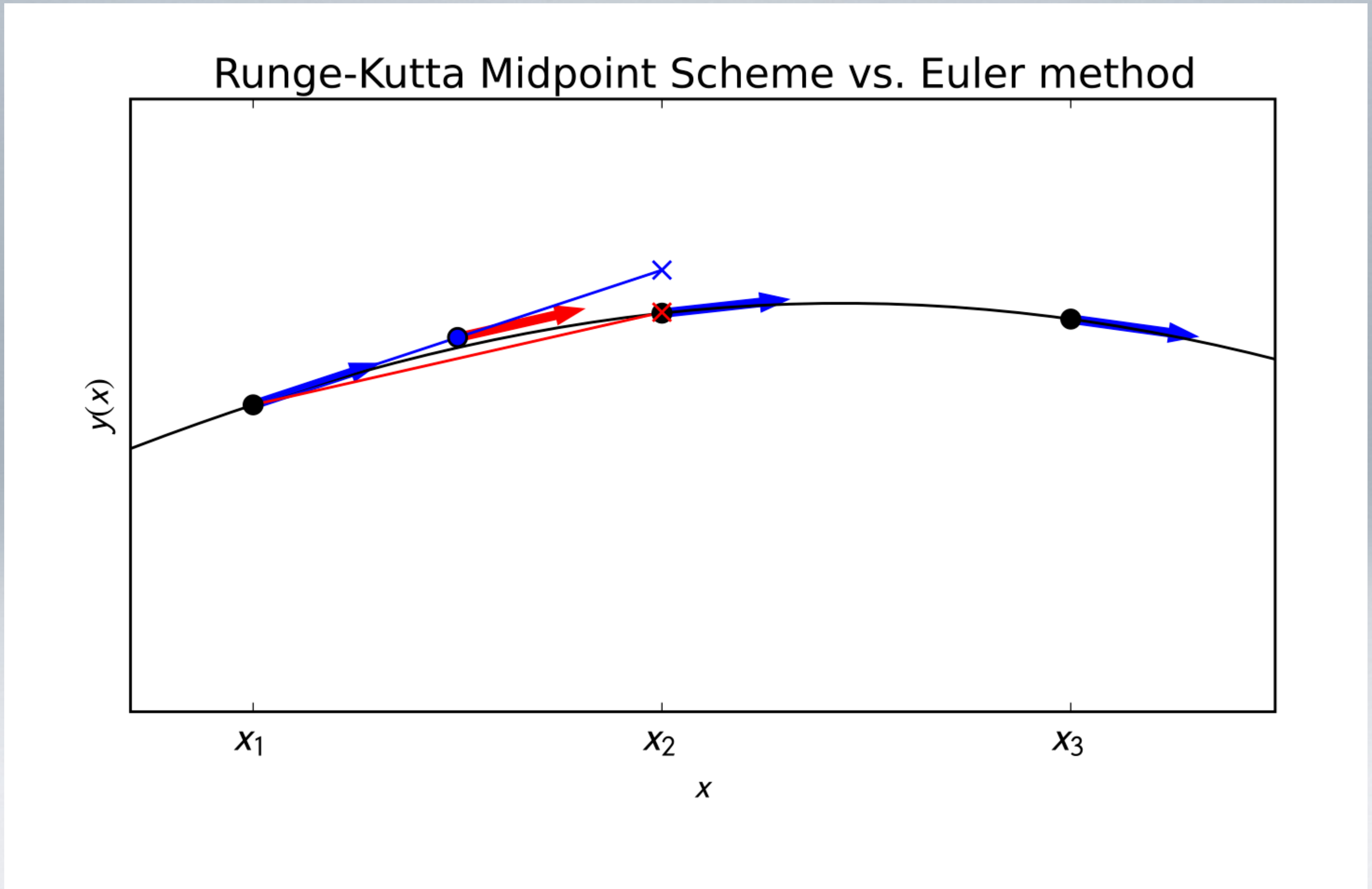
Task 2 - Simple Single Particle Motion solution 2



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Task 2 - Simple Single Particle Motion solution 2

Change the stepper method to Runge Kutta 4th order

```
def step(u, dt, p):

    f1 = derivs(u, p)
    f2 = derivs(u + dt*f1/2., p)
    f3 = derivs(u + dt*f2/2., p)
    f4 = derivs(u + dt*f3, p)
    du = dt*(f1 + 2.*f2 + 2.*f3 + f4)/6.
    u += du
    return u
```

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Task 2 - Simple Single Particle Motion solution 2

Change the plot to show the x and z components.

Change the sign and mass of the particle, and the strength of the magnetic field

Can you get a solution for a realistic situation, i.e. a proton gyrating in a magnetic field of 100 nT?

$$W_\perp = 1 \text{ keV}$$

$$q = 1.6 \times 10^{-19} \text{ C}$$

$$m_p = 1.7 \times 10^{-27} \text{ C}$$

$$\tau = 0.5 \text{ s}$$

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Task 3 - More complex Particle Motion solutions

Open and run the script tkplasma.py

How do the different E and B configurations change the particle motion in real and velocity space?

Does the velocity space result agree with expectation?

What happens to the particle's speed as it moves into a region of increasing magnetic field? What about the velocity?

Task 4 - Multiple Particle Motion solutions

Open and run the script tkplasmamulti.py

How do the different particles differ in behaviour in real and velocity space in the different magnetic field configurations?

Try different E and B fields, and vary the particle's initial speed relative to E/B

What happens if collisions are introduced? What effect does the collision frequency have on the particle behaviour?

Is there is a difference in particle distribution function between regions of different field strength? (look at the splits function)

What effect do wave particle interactions and particle absorption have?

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