

AGWAM 2020

Introduction of GR

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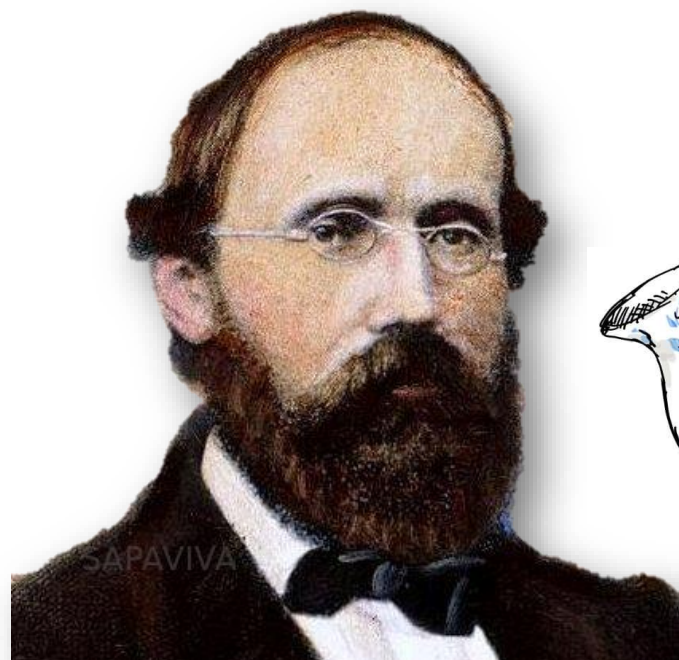
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Outline

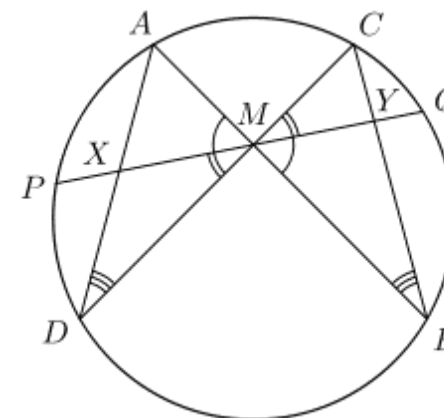
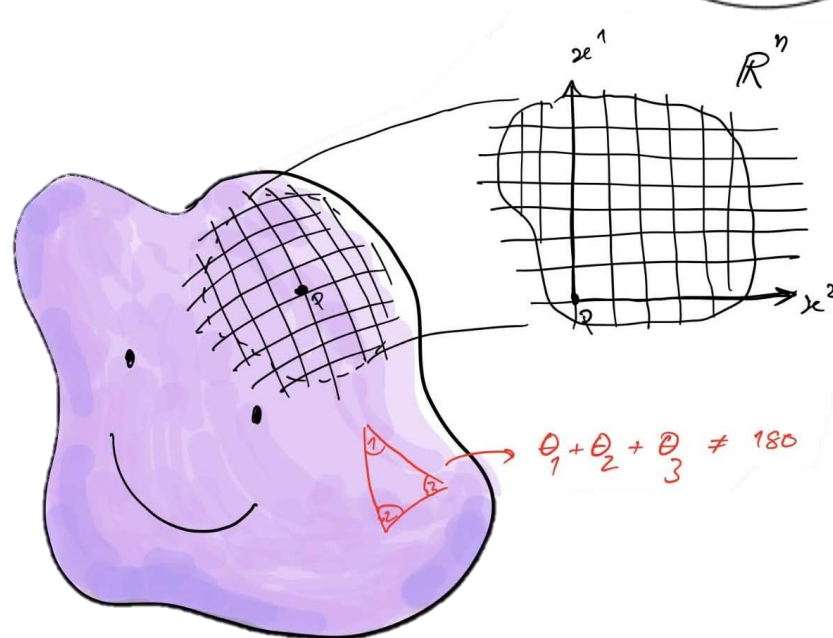
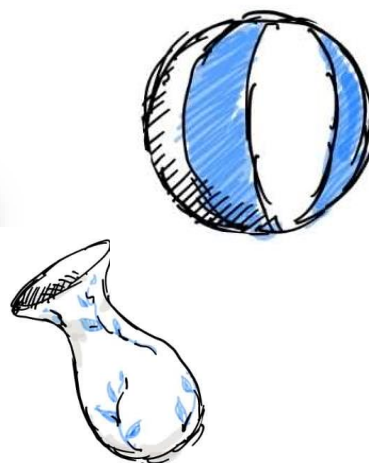
- Introduction
- Mathematical setup
 - Curves and surface
 - Manifolds
 - Distance, derivative, and curvature
- Relativity
 - Special relativity
 - General relativity
- Experimental test
 - Gravitational waves
 - New physics

Introduction:

Euclidean VS Non-euclidean geometry



(M, g)

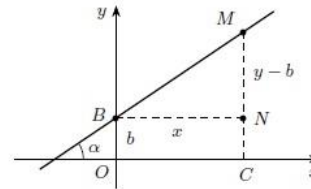
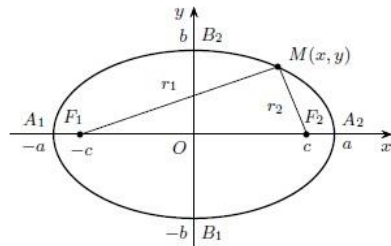
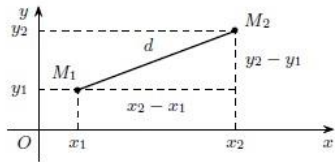


$\mathbb{R}^n$

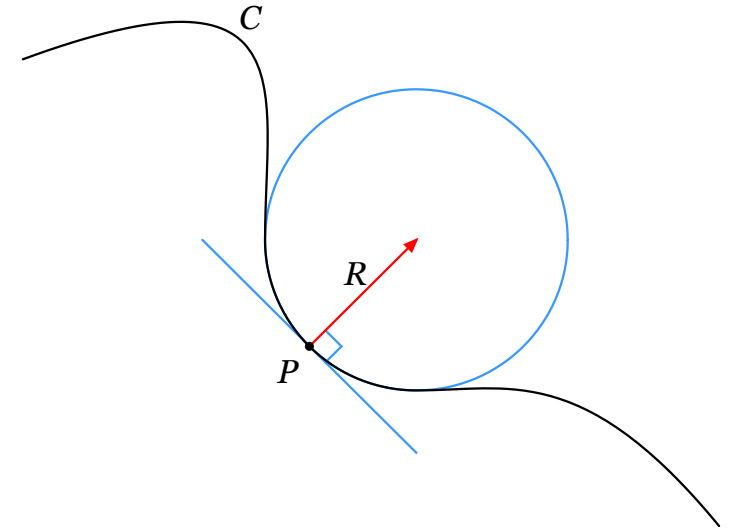
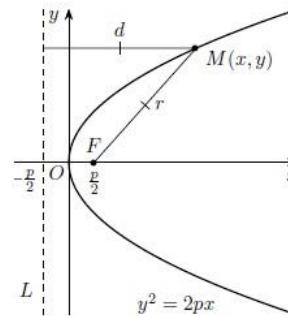
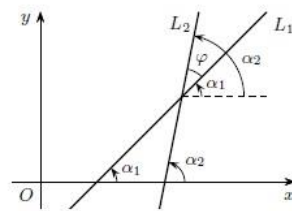
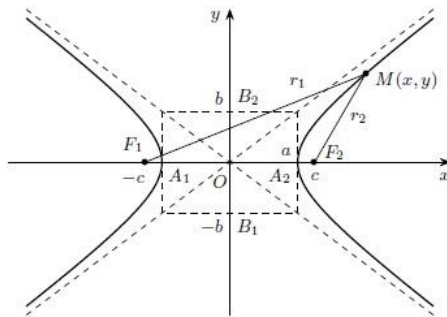
Mathematical setup: Curves and surfaces

- In the form of subspaces of Euclidean space

$$\Gamma = \{(x, y) \in \mathbb{R}^2 | F(x, y) = 0\}$$



Analytic Geometry



Curvature in 1D is the inverse of radius of circle touching the curve

$$\kappa(x) = \frac{|y''|}{(1 + y'^2)^{3/2}}$$

Mathematical setup: Curves and surfaces

- For circle

$$F(x, y) = R^2 - x^2 - y^2$$

one obtains

$$y' = -\frac{x}{y} \quad \text{and} \quad y'' = -\frac{1}{y} \left(1 + \frac{x^2}{y^2} \right)$$

$$\kappa = \frac{1}{y} \left(1 + \frac{x^2}{y^2} \right) \times \frac{1}{\left(1 + \frac{x^2}{y^2} \right)} = \frac{1}{R}$$

Exc One may try to compute the curvature for an ellipse

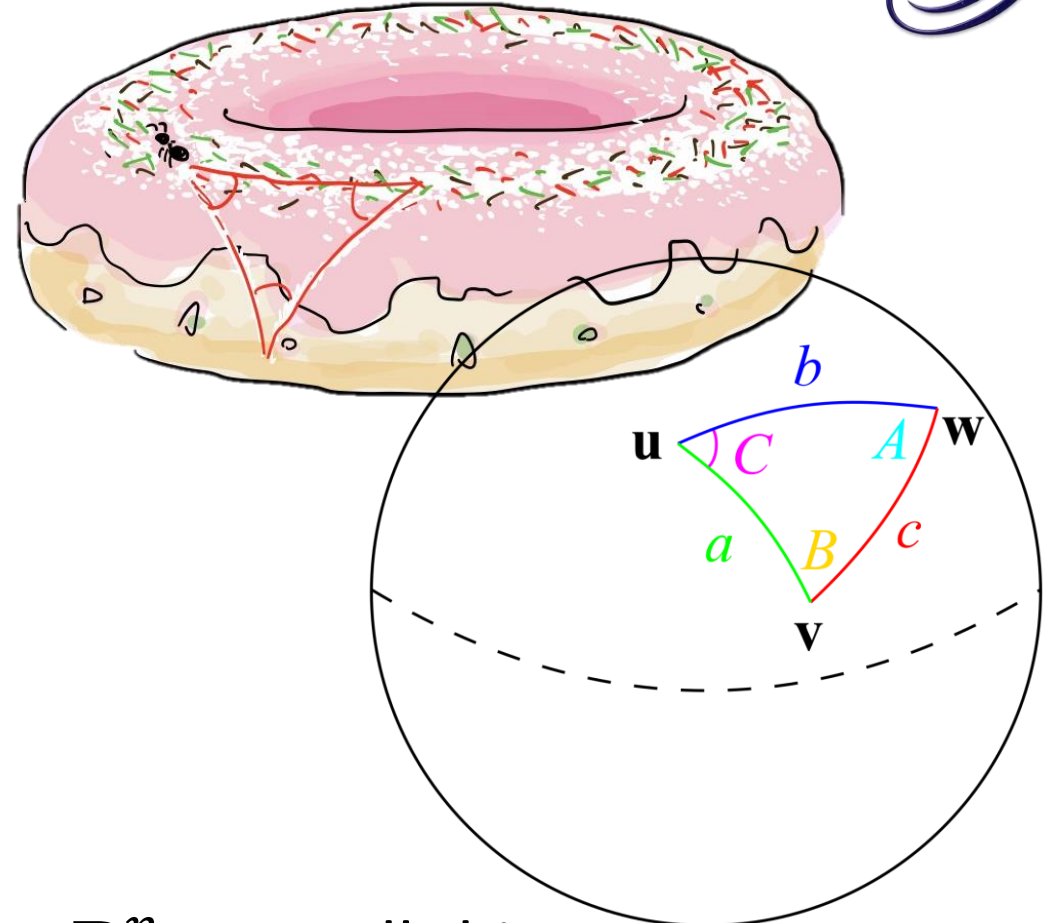
- Require additional dimension to define curvature
- Later we will know that κ is extrinsic curvature
- We will setup a machinery that allows us to define curvature intrinsically

Mathematical setup : Manifolds

What happen when you are in a curved space?

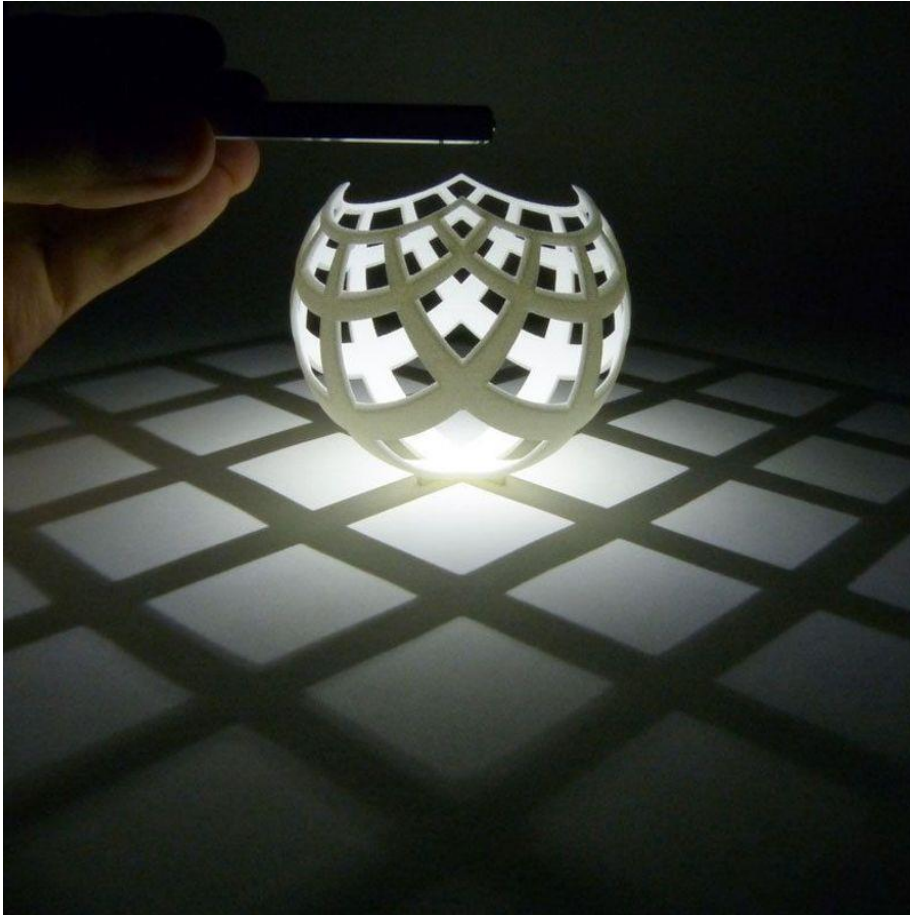
- If you draw a small enough triangle, the sum of all 3 angles will be 180°
- This probably fails when you draw a big one
- In astronomy people familiar with cosine law

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$



So in a sufficiently small area, it resembles $\mathbb{R}^n$. We call this property “locally Euclidean”.

Mathematical setup : Manifolds



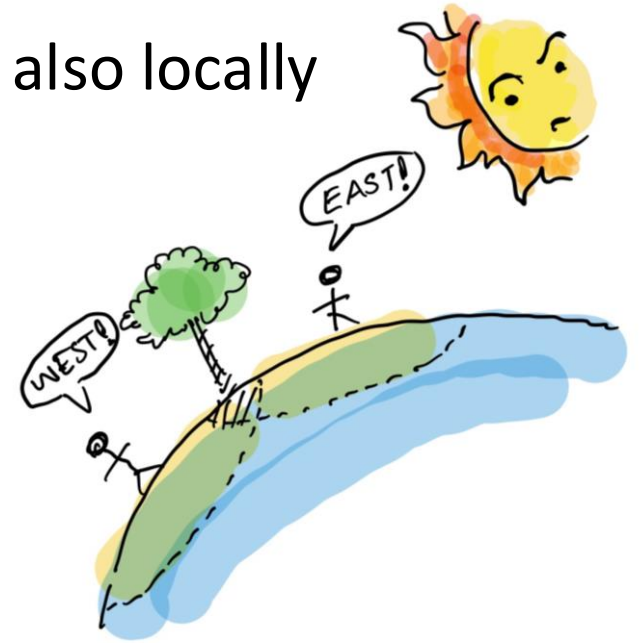
Local coordinates

- $\exists U \subset M$, s.t. the same machinery we are familiar with in Euclidean space, such as **coordinate**, can be used. The set U is call **chart**
- If your manifold is top. different from $\mathbb{R}^n \rightarrow$ need more that 1 chart
- For S^2 one needs at least 2 charts

Mathematical setup : Manifolds

Def A second countable Hausdorff topological space that is also locally Euclidean called a manifold, M

- Suppose $\{x^\mu\}$ for $\mu \in \{1, 2, \dots, n\}$ is a local coordinate around a point $p \in M$
- Another nearby point $q \in M$ has a local coordinate $\{\tilde{x}^\mu\}$
- In the overlap region, the two coordinates describe the same info one should have a coordinate transition function $\tilde{x}^\mu \mapsto x^\mu$ (and vice versa)
- This induces Jacobian matrix we know and love



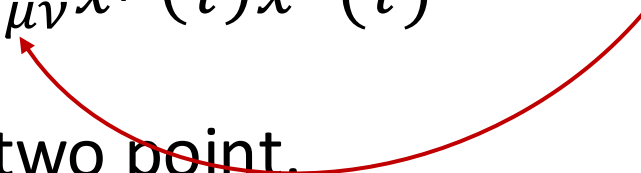
Def If every transition function is smooth, then we call M a smooth manifold

Mathematical setup: distance

- **Geodesic distance**: for a family of curve between $p, q \in M$,
 $\mathcal{C} = \{\gamma: [0,1] \rightarrow M \mid \gamma(0) = p, \gamma(1) = q\}$, the geodesic distance is

$$s = \inf_{\gamma \in \mathcal{C}} \int_{\gamma} d\tau \sqrt{g_{\mu\nu} \dot{x}^{\mu}(\tau) \dot{x}^{\nu}(\tau)}$$

Metric tensor



the shortest distance between any two point.

Exc In $\mathbb{R}^n$ the metric tensor $g = \mathbb{I}_{n \times n}$. What is the geodesic distance between two points looks like?

- **Infinitesimal distance**: Locally the distance satisfies

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

Mathematical setup: distance

- Takes $L = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ to be a Lagrangian and use the variational principle
- Geodesic equation (the coordinate of the shortest curve)

$$\frac{d^2 x^\sigma}{d\tau^2} - \Gamma_{\mu\nu}^\sigma \left(\frac{dx^\mu}{d\tau} \right) \left(\frac{dx^\nu}{d\tau} \right) = 0 ,$$

The Γ 's is called Christoffel symbol. They are numerical factors that are derived from derivatives of the metric tensor.

$$\Gamma_{\nu\mu}^\sigma = \Gamma_{\mu\nu}^\sigma = \frac{g^{\sigma\alpha}}{2} \left(\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right)$$

Exc Check that the formula is correct.

Mathematical setup: derivative

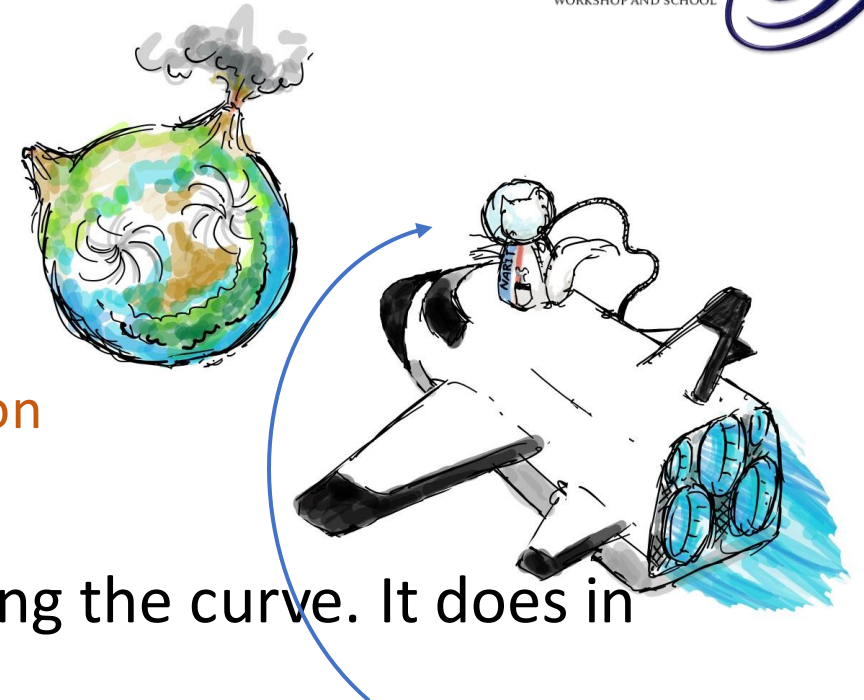
What is the meaning of geodesic equation?

- Recall that $g = \mathbb{I}_{n \times n}$ in $\mathbb{R}^n$ so that $\Gamma_{\mu\nu}^\sigma = 0$
- $\frac{d^2 x^\sigma}{d\tau^2} = \frac{dV^\sigma}{d\tau} = 0 \rightarrow$ the curve evolves without acceleration

In Euclidean the basis vector does not change along the curve. It does in curve space. Suppose Z is a vector on M

$$\begin{aligned} \frac{d(Z^\alpha e_\alpha)}{d\tau} &= \frac{dZ^\alpha}{d\tau} e_\alpha + Z^\alpha \frac{de_\alpha}{d\tau} = V^\sigma \left(\frac{\partial Z^\alpha}{\partial x^\sigma} e_\alpha + Z^\alpha \Gamma_{\sigma\alpha}^\beta e_\beta \right) \\ &= V^\sigma \left(\frac{\partial Z^\alpha}{\partial x^\sigma} + \Gamma_{\sigma\beta}^\alpha Z^\beta \right) e_\alpha \\ &= \nabla_V Z \end{aligned}$$

People call ∇ a covariant derivative (physicist), or the Levi-Civita connection (mathematician)



A cat moving along geodesic feel no acceleration

We can also rewrite geodesic equation as $\nabla_V V = 0$

Mathematical setup: derivative

Exc What is the expression for derivative of tensor

$$\nabla_V (T^{\mu\nu} e_\mu \otimes e_\nu) ?$$

Look at the formula from the previous slide.

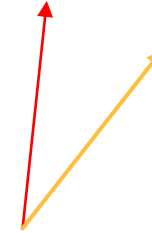
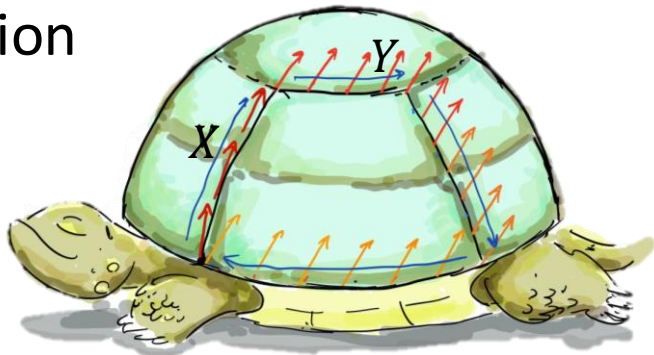
Mathematical setup: curvature

Let us try to define curvature. Suppose V, Z are vector fields

- Parallel transport (PT) wrt vector V

$$\nabla_V Z = 0$$

- Notice that: Z PT back to the starting point $\rightarrow$ change in length & direction



$$Z' = R(X, Y)Z$$

The original vector changes its length and direction after parallel transport

- Define: curvature as multi-linear map $R: TM \times TM \times \Gamma(M) \rightarrow \Gamma(M)$

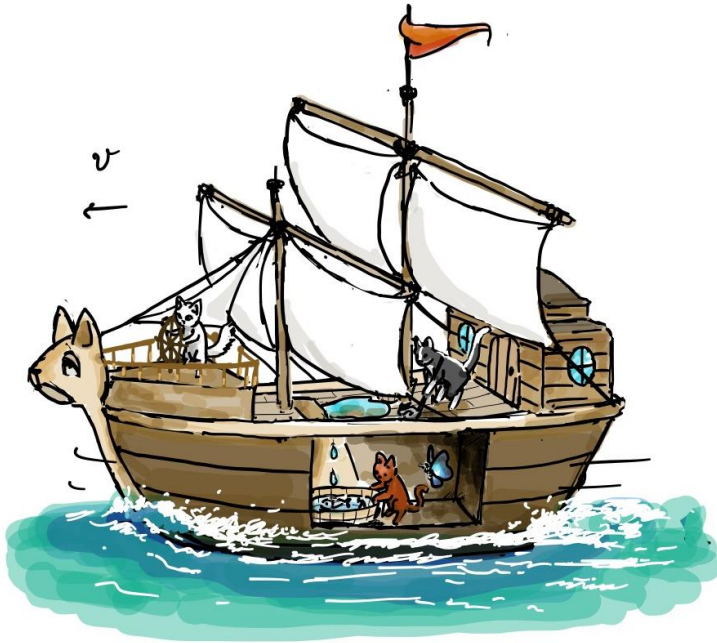
$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

- Or in coordinate basis form

$$R_{\mu\nu\rho}{}^{\sigma} = \partial_{\mu}\Gamma_{\nu\rho}^{\sigma} - \partial_{\nu}\Gamma_{\mu\rho}^{\sigma} + \Gamma_{\mu\alpha}^{\sigma}\Gamma_{\nu\rho}^{\alpha} - \Gamma_{\nu\alpha}^{\sigma}\Gamma_{\mu\rho}^{\alpha}$$

Relativity

- The law of physics are the same in all inertia frame of reference



- Equation of motion stays the same i.e. physics is same
- This is a kind of symmetry (which we will get back to it later)

Special Relativity

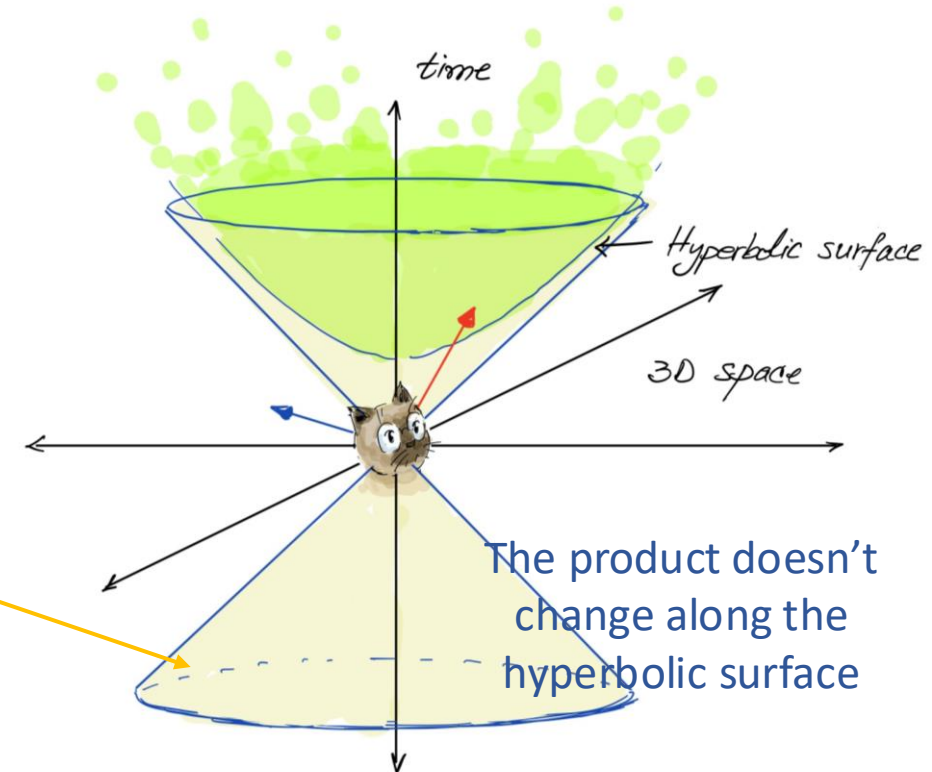
- The speed of light is the same for all observers in inertial frames
- The law of physics are the same in all inertia frame of reference

The physics is best described by 4 dim vector space $\mathbb{R}^{1,3}$ with indefinite product (Minkowski space)

$$(w, v) = w^0 v^0 - \sum_{i=1}^3 w^i v^i$$

This create the light cone for each observer. A point $x = (x^0, \vec{x}) \in \mathbb{R}^{1,3}$ from an observer

- Timelike $x^2 := (x, x) > 0$
- Lightlike $x^2 = 0$
- Spacelike $x^2 < 0$



Special Relativity: Four Vector

Many vectors quantities become 4-vector e.g.

$$p = \left(\frac{E}{c}, \vec{p} \right) \in \mathbb{R}^{1,3}$$

Einstein published the famous equation

$$E = mc^2,$$

which link the mass with energy. It was blamed for creating the nuclear bomb. If the mass is moving we get the dispersion relation.

$$E^2 - |\vec{p}|^2 c^2 = m^2 c^4 \quad \text{or} \quad p^2 = (mc)^2$$

This is one of the most important equation in any relativistic physics



General Relativity

- The effect of gravitation is indistinguishable from acceleration.

Freely falling → No acceleration



- Law of physics in freely falling frame is locally the same as in inertial frame.



- The spacetime is locally Euclidean (Minkowski)

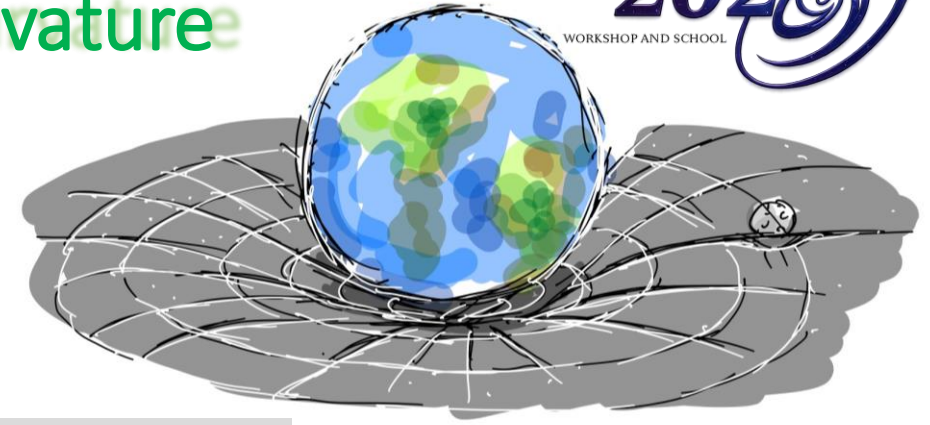


The sheep doesn't feel acceleration

General Relativity: Mass-Energy, Curvature

What create the curvature?

- Mass and energy



Tensor derived from
curvature tensor



$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$



Tensor encoding info
of mass and energy

How does Einstein come up with the equation?

- Energy momentum tensor is conserved i.e. $\nabla_\mu T^{\mu\nu} = 0$
- The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is also divergent free
- **Very difficult equation** $\Rightarrow$ No known general solution

General Relativity: Einstein-Hilbert action

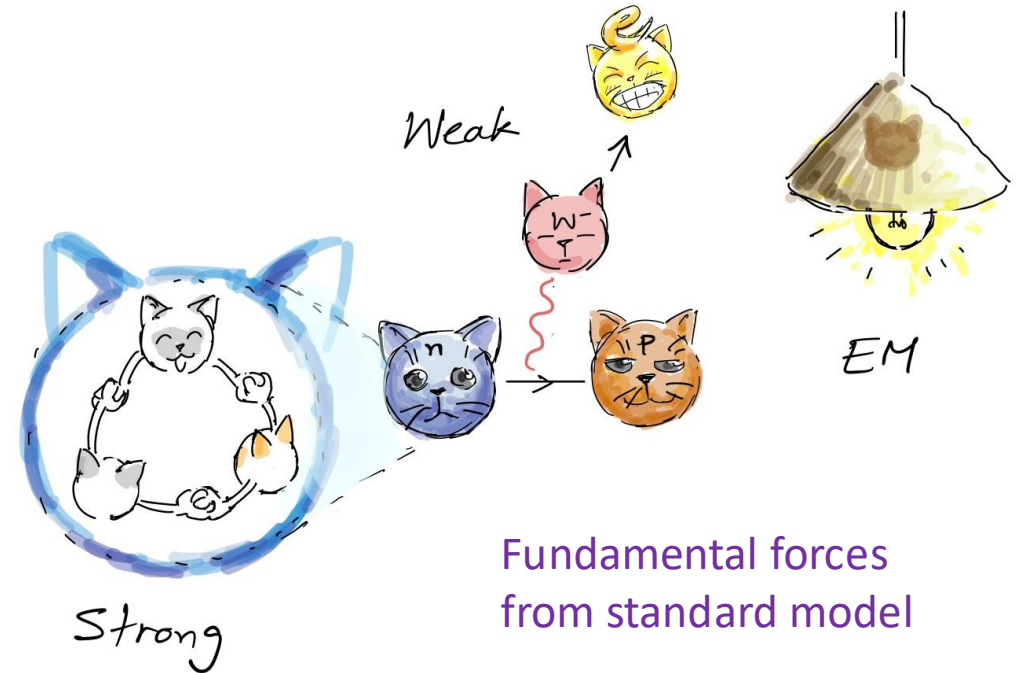
- One may use variational principle to derive Einstein equation

$$S = \int d^4x \sqrt{-g} \left(\frac{c^4}{16\pi G} R + \mathcal{L}_m \right)$$

- You get the same Einstein equation

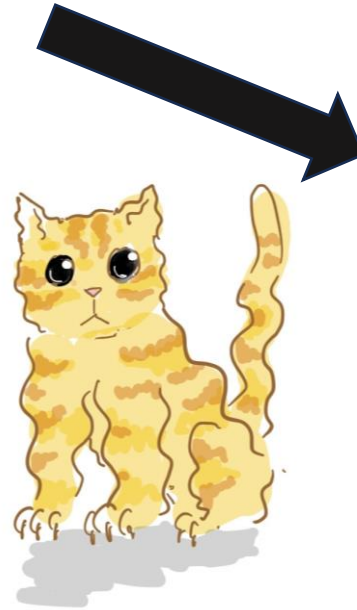
(Fun to derive Exc)

- Useful when dealing with matter field (scalar field, EM field etc.)
- Useful with modify gravity and quantum gravity



General Relativity: How do one obtains a solution??

- Solve by adding symmetry e.g. spherical, axial symmetry etc.
 - Schwarzschild solution $\rightarrow$ BH
 - Kerr solution $\rightarrow$ Astrophysical BH
 - FLRW solution $\rightarrow$ the Universe
- Approximation
 - Weak field $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$
 - Post Newtonian
- Other fun techniques
 - Algebraic classification
 - NP formalism
 - Numerical (ADM formalism)
 - Etc



General Relativity: Alternative & Beyond GR

Mond - many are ruled out by LIGO

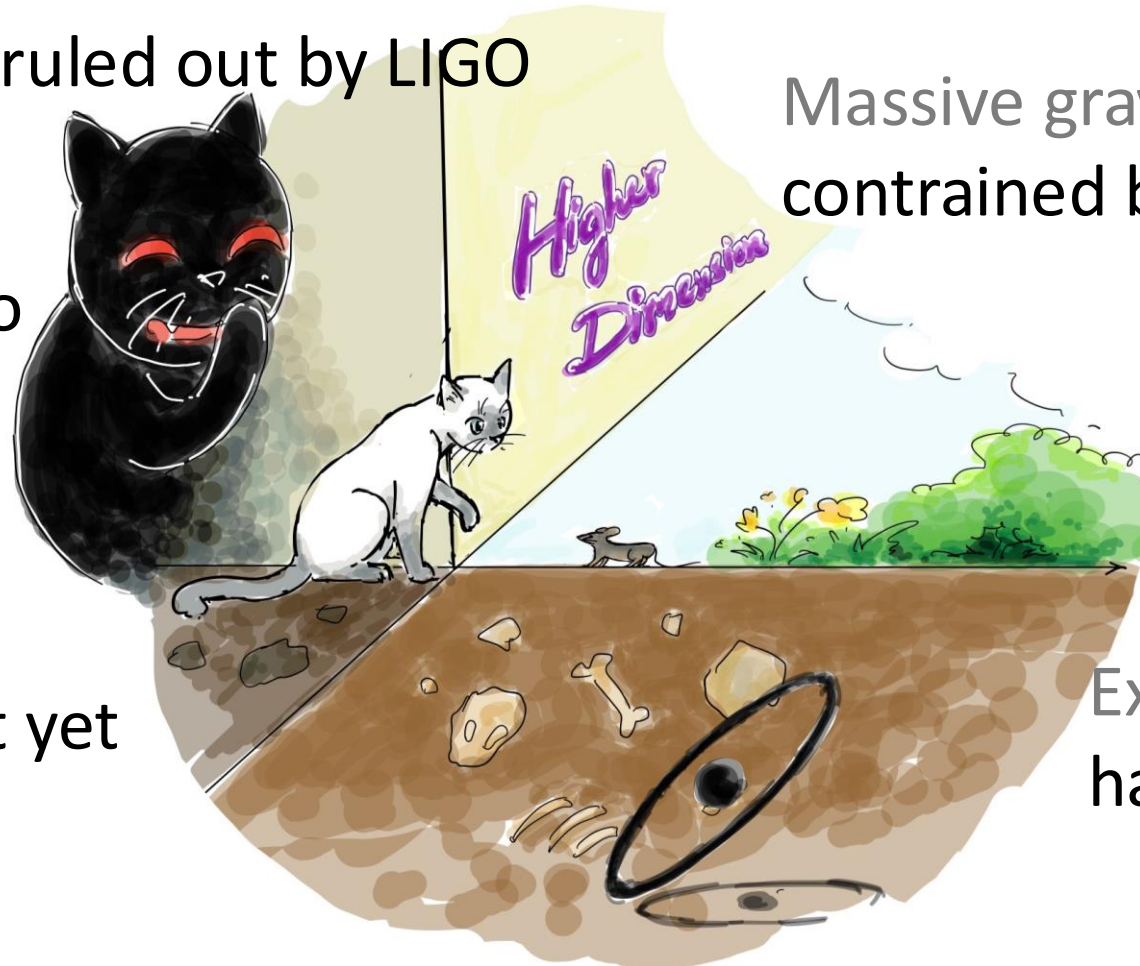
Weyl gravity – try to fix dark matter

Quantum gravity –
Dream that haven't yet
come true

Massive gravity - Mass
constrained by experiment

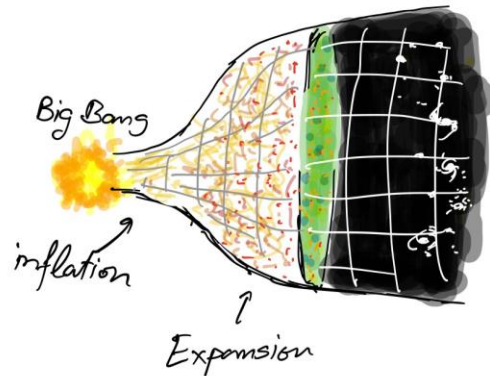
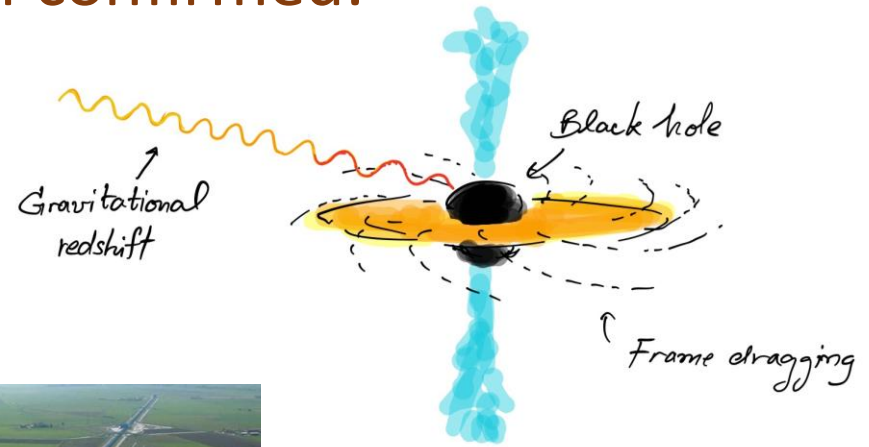
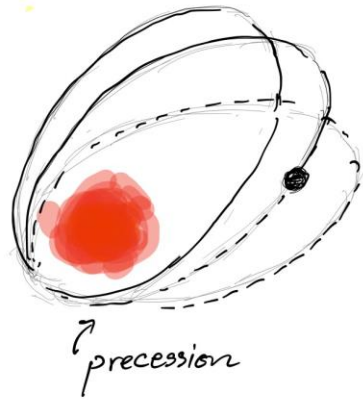
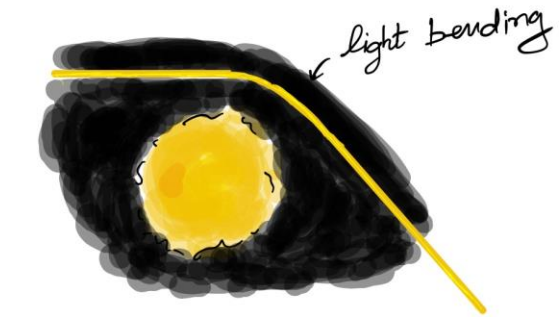
F(R) Gravity – try to
describe inflation

Extra dimension – you
have more BH solutions



Experimental test: How realistic is GR?

Predictions GR have been confirmed.



The latest discovered prediction in 2015

Experimental test: New physics

GW can give a hint of new physics

- See beyond last scattering surface
- Combine with other messenger

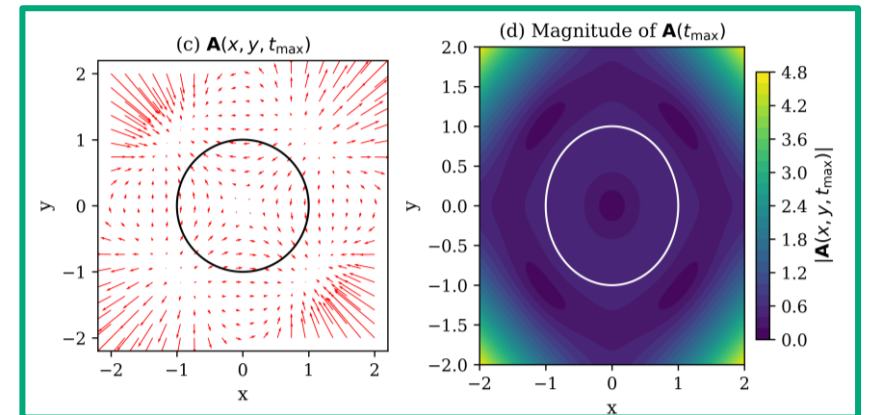
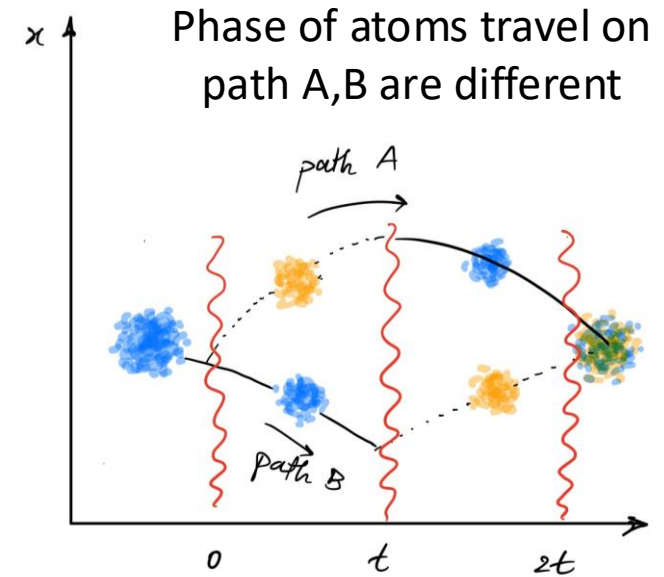
Coverage on wider frequency range

- Extreme astro events $\rightarrow$ high frequency GW
- Big astro sources $\rightarrow$ low frequency GW

Probably need new kind of detector

- Atomic detector?
- GW memories

GW induce E field if there is B.
Could it be detected?



Question?